EXACT SOLUTION OF THE SIX-VERTEX MODEL WITH DOMAIN WALL BOUNDARY CONDITIONS. DISORDERED PHASE

PAVEL M. BLEHER AND VLADIMIR V. FOKIN

Abstract. The six-vertex model, or the square ice model, with domain wall boundary conditions (DWBC) has been introduced and solved for finite N by Korepin and Izergin. The solution is based on the Yang-Baxter equations and it represents the free energy in terms of an $N\times N$ Hankel determinant. Paul Zinn-Justin observed that the Izergin-Korepin formula can be re-expressed in terms of the partition function of a random matrix model with a nonpolynomial interaction. We use this observation to obtain the large N asymptotics of the six-vertex model with DWBC in the disordered phase. The solution is based on the Riemann-Hilbert approach and the Deift-Zhou nonlinear steepest descent method. As was noticed by Kuperberg, the problem of enumeration of alternating sign matrices (the ASM problem) is a special case of the the six-vertex model. We compare the obtained exact solution of the six-vertex model with known exact results for the 1, 2, and 3 enumerations of ASMs, and also with the exact solution on the so-called free fermion line. We prove the conjecture of Zinn-Justin that the partition function of the six-vertex model with DWBC has the asymptotics, $Z_N \sim CN^{\kappa}e^{N^2 f}$ as $N\to\infty$, and we find the exact value of the exponent κ.

1. Introduction

The six-vertex model, or the model of two-dimensional ice, is stated on a square lattice with arrows on edges. The arrows obey the rule that at every vertex there are two arrows pointing in and two arrows pointing out. Such rule is sometimes called the ice-rule. There are only six possible configurations of arrows at each vertex, hence the name of the model, see Fig. 1.

We will consider the domain wall boundary conditions (DWBC), in which the arrows on the upper and lower boundaries point in the square, and the ones on the left and right boundaries point out. One possible configuration with DWBC on the 4×4 lattice is shown on Fig. 2.

The name of the square ice comes from the two-dimensional arrangement of water molecules, H_2O, with oxygen atoms at the vertices of the lattice and one hydrogen atom between each pair of adjacent oxygen atoms. We place an arrow in the direction from a hydrogen atom toward an oxygen atom if there is a bond between them. Thus, as we already noticed before, there are two inbound and two outbound arrows at each vertex.

Date: February 7, 2008.

The first author is supported in part by the National Science Foundation (NSF) Grant DMS-0354962.

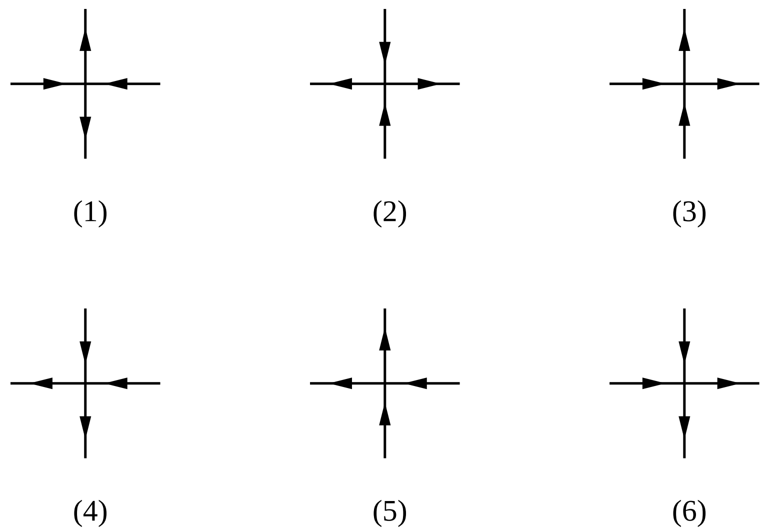

FIGURE 1. The six arrow configurations allowed at a vertex.

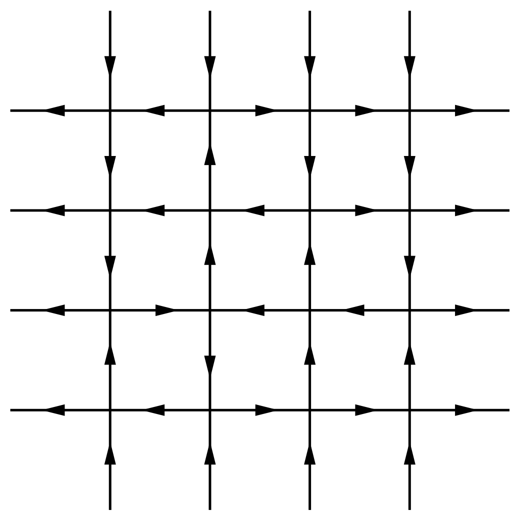

FIGURE 2. An example of 4×4 configuration.

For each possible vertex state we assign a weight w_i, $i = 1, \dots, 6$, and define, as usual, the partition function, as a sum over all possible arrow configurations of the product of the vertex weights,

$$Z_N = \sum_{\text{arrow configurations}} \prod_{i=1}^{6} w_i^{n_i}, \tag{1.1}$$

where n_i is the number of vertices in the state i in a given arrow configuration. We will consider the case, when the weights are invariant under the simultaneous reversal of all arrows, i.e.,

$$a := w_3 = w_4, \quad b := w_5 = w_6, \quad c := w_1 = w_2. \tag{1.2}$$

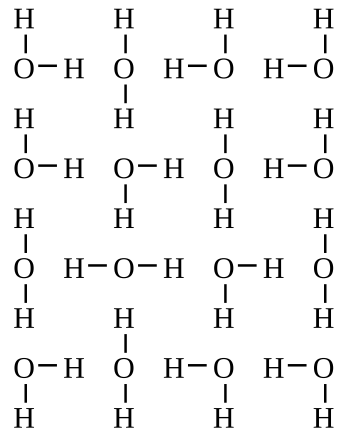

Figure 3. The corresponding ice model.

Define the parameter Δ as

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}. \tag{1.3}$$

There are three physical phases for the six-vertex model: the ferroelectric phase, $\Delta > 1$; the anti-ferroelectric phase, $\Delta < -1$; and, the disordered phase, $-1 < \Delta < 1$. The phase diagram of the model is shown on Fig. 4.

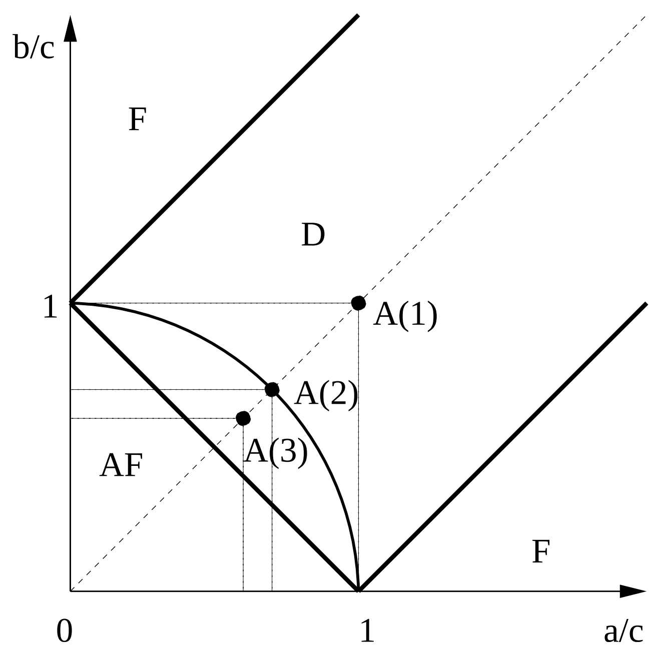

Figure 4. The phase diagram of the model, where **F**, **AF** and **D** mark ferroelectric, antiferroelectric, and disordered phases, respectively. The circular arc corresponds to the so-called "free fermion" line, where $\Delta = 0$, and the three dots correspond to 1-, 2-, and 3-enumeration of alternating sign matrices.

In these phases we parametrize the weights in the standard way: for the ferroelectric phase,

$$a = \sinh(t-\gamma), \quad b = \sinh(t+\gamma), \quad c = \sinh(2\gamma), \quad |\gamma| < t, \tag{1.4}$$

for the anti-ferroelectric phase,

$$a = \sinh(\gamma-t), \quad b = \sinh(\gamma+t), \quad c = \sinh(2\gamma), \quad |t| < \gamma, \tag{1.5}$$

and for the disordered phase,

$$a = \sin(\gamma-t), \quad b = \sin(\gamma+t), \quad c = \sin(2\gamma), \quad |t| < \gamma. \tag{1.6}$$

Here we will discuss the disordered phase, and we will use parametrization (1.6).

A solution for the free energy of the six-vertex model with periodic boundary conditions (PBC) was found by Lieb [17]–[20] by means of the Bethe Ansatz. In the most general form of the six-vertex model the Bethe Ansatz solution with PBC was obtained by Sutherland [26]. A detailed classification of the phases of the model is given in the review paper of Lieb and Wu [21]; see also the book of Baxter [1]. The six-vertex model with antiperiodic boundary conditions is solved in [2].

The six-vertex model with DWBC was introduced by Korepin in [13], who derived an important recursion relation for the partition function of the model. This lead to a beautiful determinantal formula of Izergin [11], for the partition function of the six-vertex model with DWBC. A detailed proof of this formula and its generalizations are given in the paper of Izergin, Coker, and Korepin [12]. When the weights are parameterized according to (1.6), the formula of Izergin is

$$Z_N = \frac{[\sin(\gamma+t)\sin(\gamma-t)]^{N^2}}{\left(\prod_{n=0}^{N-1} n!\right)^2}\,\tau_N\,, \tag{1.7}$$

where τ_N is the Hankel determinant,

$$\tau_N = \det\left(\frac{d^{i+k-2}\phi}{dt^{i+k-2}}\right)_{1\le i,k\le N}, \tag{1.8}$$

and

$$\phi(t) = \frac{\sin(2\gamma)}{\sin(\gamma+t)\sin(\gamma-t)}\,. \tag{1.9}$$

An elegant derivation of the Izergin determinantal formula from the Yang-Baxter equations is given in the papers of Korepin and Zinn-Justin [14] and Kuperberg [16].

One of the applications of the determinantal formula is that it implies that the partition function τ_N solves the Toda equation,

$$\tau_N\tau_N'' - {\tau_N'}^2 = \tau_{N+1}\tau_{N-1}, \qquad N \ge 1, \qquad (') = \frac{\partial}{\partial t}\,, \tag{1.10}$$

cf. [24]. This was used by Korepin and Zinn-Justin [14] to derive the free energy of the six-vertex model with DWBC, assuming some Ansatz on the behavior of subdominant terms in the large N asymptotics of the free energy.

Another application of the Izergin determinantal formula is that τ_N can be expressed in terms of a partition function of a random matrix model. The relation to the random matrix model was obtained and used by Zinn-Justin [28]. This relation will be very important for us. It can be derived as follows. For the evaluation of the Hankel determinant, it is convenient

to use the integral representation of $\phi(t)$, namely, to write it in the form of the Laplace transform,

$$\phi(t) = \int_{-\infty}^{\infty} e^{t\lambda} m(\lambda) d\lambda, \tag{1.11}$$

where

$$m(\lambda) = \frac{\sinh \frac{\lambda}{2}(\pi - 2\gamma)}{\sinh \frac{\lambda}{2}\pi}\,. \tag{1.12}$$

Then

$$\frac{d^i \phi}{dt^i} = \int_{-\infty}^{\infty} \lambda^i e^{t\lambda} m(\lambda) d\lambda, \tag{1.13}$$

and by substituting this into the Hankel determinant, (1.8), we obtain that

$$\begin{aligned} \tau_N &= \int \prod_{i=1}^{N} [e^{t\lambda_i} m(\lambda_i) d\lambda_i] \det(\lambda_i^{i+k-2})_{1\le i,k\le N} \\ &= \int \prod_{i=1}^{N} [e^{t\lambda_i} m(\lambda_i) d\lambda_i] \det(\lambda_i^{k-1})_{1\le i,k\le N} \prod_{i=1}^{N} \lambda_i^{i-1}. \end{aligned} \tag{1.14}$$

Consider any permutation $\sigma \in S_N$ of variables λ_i. From the last equation we have that

$$\tau_N = \int \prod_{i=1}^{N} [e^{t\lambda_i} m(\lambda_i) d\lambda_i] (-1)^{\sigma} \det(\lambda_i^{k-1})_{1\le i,k\le N} \prod_{i=1}^{N} \lambda_{\sigma(i)}^{i-1}. \tag{1.15}$$

By summing over $\sigma \in S_N$, we obtain that

$$\tau_N = \frac{1}{N!} \int \prod_{i=1}^{N} [e^{t\lambda_i} m(\lambda_i) d\lambda_i] \Delta(\lambda)^2 \tag{1.16}$$

(see [28]), where $\Delta(\lambda)$ is the Vandermonde determinant,

$$\Delta(\lambda) = \det(\lambda_i^{k-1})_{1\le i,k\le N} = \prod_{i<k} (\lambda_k - \lambda_i). \tag{1.17}$$

Equation (1.16) expresses τ_N in terms of a matrix model integral. Namely, if $m(x) = e^{-V(x)}$, then

$$\tau_N = \frac{\prod_{n=0}^{N-1} n!}{\pi^{N(N-1)/2}} \int dM e^{\mathrm{Tr}[tM - V(M)]}, \tag{1.18}$$

where the integration is over the space of $N \times N$ Hermitian matrices. The matrix model integral can be solved, furthermore, in terms of orthogonal polynomials.

Introduce monic polynomials $P_n(x) = x^n + \dots$ orthogonal on the line with respect to the weight $e^{tx} m(x)$, so that

$$\int_{-\infty}^{\infty} P_n(x) P_m(x) e^{tx} m(x) dx = h_n \delta_{nm}. \tag{1.19}$$

Then it follows from (1.16) that

$$\tau_N = \prod_{n=0}^{N-1} h_n. \tag{1.20}$$

The orthogonal polynomials satisfy the three term recurrent relation,

$$xP_n(x) = P_{n+1}(x) + Q_n P_n(x) + R_n P_{n-1}(x), \tag{1.21}$$

where R_n can be found as

$$R_n = \frac{h_n}{h_{n-1}}\,, \tag{1.22}$$

see, e.g., [25]. This gives that

$$h_n = h_0 \prod_{j=1}^{n} R_j, \tag{1.23}$$

where

$$h_0 = \int_{-\infty}^{\infty} e^{tx} m(x)dx = \frac{\sin(2\gamma)}{\sin(\gamma+t)\sin(\gamma-t)}\,. \tag{1.24}$$

By substituting (1.23) into (1.20), we obtain that

$$\tau_N = h_0^N \prod_{n=1}^{N-1} R_n^{N-n}. \tag{1.25}$$

We will prove the following asymptotics of the recurrent coefficients R_n.

Theorem 1.1. *As $n\to\infty$,*

$$R_n = \frac{n^2}{\gamma^2}\left[R + \cos(n\omega) \sum_{j:\,\kappa_j\le 2} c_j n^{-\kappa_j} + cn^{-2} + O(n^{-2-\varepsilon})\right], \qquad \varepsilon>0, \tag{1.26}$$

where the sum is finite and it goes over $j=1,2,\ldots$ such that $\kappa_j\le 2$,

$$R = \left(\frac{\pi}{2\cos\frac{\pi\zeta}{2}}\right)^2, \quad \zeta \equiv \frac{t}{\gamma}\,; \qquad \omega = \pi(1+\zeta)\,; \qquad \kappa_j = 1 + \frac{2j}{\frac{\pi}{2\gamma}-1}\,, \tag{1.27}$$

and

$$c_j = \frac{2\gamma e^{\varphi(y_j)}}{\cos\frac{\pi\zeta}{2}}(-1)^j \sin\frac{\pi j}{1-\frac{2\gamma}{\pi}}\,, \tag{1.28}$$

where

$$y_j = \frac{\pi j}{\frac{\pi}{2\gamma}-1}\,, \tag{1.29}$$

and

$$\varphi(y) = -\frac{2y}{\pi}\ln\left(2\pi\cos\frac{\pi\zeta}{2}\right) + \frac{2}{\pi}\left[\int_0^{\infty} \arg(\mu+iy) f(\mu)d\mu + y\ln y - y\right], \tag{1.30}$$

where

$$f(\mu) = \frac{\pi}{2\gamma}\coth\mu\frac{\pi}{2\gamma} - \left(\frac{\pi}{2\gamma}-1\right)\coth\mu\left(\frac{\pi}{2\gamma}-1\right) - \operatorname{sgn}\mu. \tag{1.31}$$

Also,

$$c = \frac{\pi\gamma^2}{6(\pi-2\gamma)\cos^2\frac{\pi\zeta}{2}} - \frac{\pi^2}{48\cos^2\frac{\pi\zeta}{2}}\,. \tag{1.32}$$

The error term in (1.26) is uniform on any compact subset of the set

$$\left\{(t,\gamma):\ |t|<\gamma,\ 0<\gamma<\frac{\pi}{2}\right\}. \tag{1.33}$$

Remark. The method of the proof allows an extension of formula (1.26) to an asymptotic series in negative powers of n. We stopped at terms of the order of n^{-2}, because for higher order terms formula for c_j becomes complex.

Denote

$$F_N = \frac{1}{N^2}\ln\frac{\tau_N}{\left(\prod_{n=0}^{N-1} n!\right)^2}. \tag{1.34}$$

From Theorem 1.1 we will derive the following result.

Theorem 1.2. *As $N\to\infty$,*

$$F_N = F + O(N^{-1}), \tag{1.35}$$

where

$$F = \frac{1}{2}\ln\frac{R}{\gamma^2} = \ln\frac{\pi}{2\gamma\cos\frac{\pi\zeta}{2}}. \tag{1.36}$$

This coincides with the formula of work [28], obtained in the saddle-point approximation. Earlier it was derived in work [14], from some Ansatz for the free energy asymptotics. For the partition function Z_N in (1.7) we obtain from Theorem 1.2 the formula,

$$\frac{1}{N^2}\ln Z_N = f + O(N^{-1}) \qquad f = \ln\left(\frac{\pi[\cos(2t)-\cos(2\gamma)]}{4\gamma\cos\frac{\pi t}{2\gamma}}\right). \tag{1.37}$$

Let us compare this formula and asymptotics (1.26) with known exact results.

The free fermion line, $\gamma=\frac{\pi}{4}$, $|t|<\frac{\pi}{4}$. In this case the exact result is

$$Z_N = 1, \tag{1.38}$$

see, e.g., [7], which implies $f=0$. This agrees with formula (1.37), which also gives $f=0$ when $\gamma=\frac{\pi}{4}$. Moreover, the orthogonal polynomials in this case are the Meixner-Pollaczek polynomials, for which

$$R_n = \frac{4n^2}{\cos^2 2t} = \frac{n^2 R}{\gamma^2}, \tag{1.39}$$

cf. [7]. Thus, formula (1.26) is exact on the free fermion line, with no error term. This agrees with Theorem 1.1, because from (1.28), (1.32), $c_j=c=0$ when $\gamma=\frac{\pi}{4}$.

The ASM (ice) point, $\gamma=\frac{\pi}{3}$, $t=0$. In this case we obtain from (1.6) that

$$a=b=c=\frac{\sqrt{3}}{2}, \tag{1.40}$$

hence

$$Z_N = \left(\frac{\sqrt{3}}{2}\right)^{N^2} A(N), \tag{1.41}$$

where $A(N)$ is the number of configurations in the six-vertex model with DWBC. There is a one-to-one correspondence between the set of configurations in the six-vertex model with DWBC and the set of $N\times N$ alternating sign matrices. By definition, an alternating sign matrix (ASM) is a matrix with the following properties:

- all entries of the matrix are $-1, 0, 1$;
- if we look at the sequence of (-1)'s and 1's, they are alternating along any row and any column;
- the sum of entries is equal to 1 along any row and any column.

The above correspondence is established as follows: given a configuration of arrows on edges, we assign (-1) to any vertex of type (1) on Fig. 1, 1 to any vertex of type (2), and 0 to any vertex of other types. Then the configuration on the vertices gives rise to an ASM, and this correspondence is one-to-one. For instance, Fig. 5 shows the ASM corresponding to the configuration of arrows on Fig. 2.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

FIGURE 5. ASM for the configuration of Fig. 2.

For the number of ASMs there is an exact formula:

$$A(N) = \prod_{n=0}^{N-1} \frac{(3n+1)!n!}{(2n)!(2n+1)!}\,. \tag{1.42}$$

This formula was conjectured in [22], [23], and proved by Zeilberger [27] by combinatorial arguments. Another proof was given by Kuperberg [16], who used formula (1.7). The relation to classical orthogonal polynomials was found by Colomo and Pronko [7], who used this relation to give a new proof of the ASM conjecture. The orthogonal polynomials in this case are the continuous Hahn polynomials and from [7] we find that

$$R_n = \frac{n^2(9n^2-1)}{4n^2-1} = \frac{9n^2}{4} + \frac{5}{16} + O(n^{-2}). \tag{1.43}$$

Formula (1.26) gives

$$R_n = \frac{9n^2}{\pi^2}\left[\frac{\pi^2}{4} + \frac{5\pi^2}{144n^2} + O(n^{-2-\varepsilon})\right], \tag{1.44}$$

which agrees with (1.43). From (1.42) we find, see Appendix A, that as $N \to \infty$,

$$A(N) = C\left(\frac{3\sqrt{3}}{4}\right)^{N^2} N^{-\frac{5}{36}}\left(1 - \frac{115}{15552N^2} + O(N^{-3})\right), \tag{1.45}$$

where $C > 0$ is a constant, so that

$$Z_N = C\left(\frac{9}{8}\right)^{N^2} N^{-\frac{5}{36}}\left(1 - \frac{115}{15552N^2} + O(N^{-3})\right), \qquad N \to \infty. \tag{1.46}$$

Formula (1.37) gives $f = \ln \frac{9}{8}$, which agrees with the last formula.

The $x = 3$ ASM point, $\gamma = \frac{\pi}{6}$, $t = 0$. Here the exact result is

$$Z_N = \frac{3^{N/2}}{2^{N^2}} A(N;3), \tag{1.47}$$

where

$$\begin{cases} A(2m+1;3) = 3^{m(m+1)} \displaystyle\prod_{k=1}^{m} \left[\frac{(3k-1)!}{(m+k)!}\right]^2, \\ A(2m+2;3) = 3^m \dfrac{(3m+2)!m!}{[(2m+1)!]^2} A(2m+1;3)\,. \end{cases} \tag{1.48}$$

In this case $A(N;3)$ counts the number of alternating sign matrices with weight 3^k, where k is the number of (-1) entries. Formula (1.48) for $A(N;3)$ was conjectured in [22], [23] and proved in [16]. The relation to classical orthogonal polynomials was again found by Colomo and Pronko [7], who used it to give a new proof of formula (1.48) for the 3-enumeration of ASMs. The orthogonal polynomials in this case are expressed in terms of the continuous dual Hahn polynomials and from [7] we find that

$$R_{2m} = 36m^2, \qquad R_{2m+1} = 4(3m+1)(3m+2)\,. \tag{1.49}$$

In this case the subdominant term in the asymptotics of R_n exhibits a period 2 oscillation. Namely, we obtain from the last formula that

$$R_n = 9n^2 + \frac{-1+(-1)^n}{2}\,. \tag{1.50}$$

This perfectly fits to the frequency value $\omega = \pi$ for $\zeta = 0$ in (1.27). Moreover, formula (1.26) gives

$$R_n = \frac{36n^2}{\pi^2}\left[\frac{\pi^2}{4} + \frac{(-1)^n c_1}{n^2} - \frac{\pi^2}{72n^2} + O(n^{-2-\varepsilon})\right], \tag{1.51}$$

which agrees with (1.50) and it provides with the value of $c_1 = \frac{\pi^2}{72}$.

From (1.48) we find, see Appendix A, that as $m \to \infty$,

$$A(2m;3) = C_3 \left(\frac{3}{2}\right)^{4m^2} 3^{-m}(2m)^{\frac{1}{18}} \left(1 + \frac{77}{7776m^2} + O(N^{-3})\right), \tag{1.52}$$

where $C_3 > 0$ is a constant, and

$$A(2m+1;3) = C_3 \left(\frac{3}{2}\right)^{(2m+1)^2} 3^{-\frac{2m+1}{2}}(2m+1)^{\frac{1}{18}} \left(1 + \frac{131}{7776m^2} + O(m^{-3})\right). \tag{1.53}$$

so that

$$A(N;3) = C_3 \left(\frac{3}{2}\right)^{N^2} 3^{-\frac{N}{2}} N^{\frac{1}{18}} \left(1 + \frac{104 - 27(-1)^N}{1944N^2} + O(N^{-3})\right), \tag{1.54}$$

and

$$Z_N = C_3 \left(\frac{3}{4}\right)^{N^2} N^{\frac{1}{18}} \left(1 + \frac{104 - 27(-1)^N}{1944N^2} + O(N^{-3})\right), \qquad N \to \infty. \tag{1.55}$$

Formula (1.37) gives $f = \ln \frac{3}{4}$, which agrees with the last formula.

We have the identity,

$$\frac{\partial^2 F_N}{\partial t^2} = \frac{R_N}{N^2}\,, \tag{1.56}$$

see, e.g., [5], which is equivalent to the Toda equation (1.10). By using identity (1.56), we obtain from Theorem 1.1 the following asymptotics.

Theorem 1.3. *As $N \to \infty$,*

$$\frac{\partial^2 (F_N - F)}{\partial t^2} = \frac{1}{\gamma^2}\cos(N\omega)\sum_{j:\,\kappa_j\le 2} c_j N^{-\kappa_j} + cN^{-2} + O(N^{-2-\varepsilon}). \tag{1.57}$$

This gives a quasiperiodic asymptotics, as $N \to \infty$, of the second derivative of the subdominant terms.

Zinn-Justin's conjecture. Paul Zinn-Justin conjectured in [28] that

$$Z_N \sim CN^\kappa e^{N^2 f}\,, \tag{1.58}$$

i.e.,

$$\lim_{N\to\infty} \frac{Z_N}{CN^\kappa e^{N^2 f}} = 1\,. \tag{1.59}$$

Formulae (1.38), (1.46), and (1.55) confirm this conjecture, with the value of κ given as

$$\kappa = \begin{cases} 0, \quad \gamma = \dfrac{\pi}{4}, \quad |t| < \dfrac{\pi}{4}\,; \\ -\dfrac{5}{36}, \quad \gamma = \dfrac{\pi}{3}, \quad t = 0; \\ \dfrac{1}{18}, \quad \gamma = \dfrac{\pi}{6}, \quad t = 0. \end{cases} \tag{1.60}$$

Bogoliubov, Kitaev and Zvonarev obtained in [6] the asymptotics of Z_N on the line $\frac{a}{c} + \frac{b}{c} = 1$, separating the disordered and antiferroelectric phases. This corresponds to the value $\gamma = 0$. They found that in this case formula (1.58) holds with $\kappa = \frac{1}{12}$.

With the help of Theorem 1.1 we will prove the following result.

Theorem 1.4. *We have that*

$$Z_N = CN^\kappa e^{N^2 f}\left(1 + O(N^{-\varepsilon})\right), \qquad \varepsilon > 0, \tag{1.61}$$

where

$$\kappa = \frac{1}{12} - \frac{2\gamma^2}{3\pi(\pi - 2\gamma)}\,, \tag{1.62}$$

and $C > 0$ is a constant.

This proves the conjecture of Zinn-Justin, and it gives the exact value of the exponent κ. Let us remark, that the presence of the power-like factor N^κ in the asymptotics of Z_N in (1.61) is rather unusual from the point of view of random matrix models. As was proven rigorously by Ercolani and McLaughlin [10], in the one-matrix model with an independent of N analytic interaction $V(M) = M^2 + tV_1(M)$, where $t > 0$ is small, $N^{-2}\ln\frac{Z_N(t)}{Z_N(0)}$ is expanded into an asymptotic series in powers of N^{-2}.

The set-up for the remainder of the paper is the following:

- In Section 2 we describe a rescaling of the weight, which was introduced by Zinn-Justin [28], and which is convenient in the subsequent calculations. The rescaled random matrix model is described by a potential $V_N(x)$ such that as $N \to \infty$, it approaches a limiting potential $V(x)$.
- In Sections 3-5 we evaluate the equilibrium measures for the random matrix models, first for the limiting one, corresponding to $V(x)$, and then for the random matrix model, which corresponds to $V_N(x)$.
- In Section 6 we remind the Riemann-Hilbert problem for orthogonal polynomials, and in Sections 7-9 we carry out the large N asymptotic analysis of the Riemann-Hilbert problem, via a sequence of transformations and the Deift-Zhou nonlinear steepest descent method.
- We use the results of this analysis in Section 10, where we obtain the large N asymptotics of the recurrent coefficients and prove Theorem 1.1. The central point in the derivation of the subdominant asymptotics of the recurrent coefficient is a deformation of the lenses boundary, see Section 1.1. During this deformation, the lenses boundary crosses poles of the function $e^{-NG_N(z)}$, and every time it crosses a pole, a new subdominant term arises in the asymptotics of the recurrent coefficient. Section 11 gives a proof to Theorems 1.2-1.4.
- Finally, there are several Appendices to the paper, where some auxiliary results are proved and some exact large N asymptotics are obtained.

2. Rescaling of the Weight

Following [28], let us substitute $\lambda_i = \frac{N\mu_i}{\gamma}$ in (1.16). This reduces τ_N to

$$\tau_N = \frac{N^{N^2}\tilde{\tau}_N}{N!\gamma^{N^2}}, \tag{2.1}$$

where

$$\tilde{\tau}_N = \int \prod_{i=1}^{N} \left[e^{N\zeta\mu_i} m\left(\frac{N\mu_i}{\gamma}\right) d\mu_i \right] \Delta(\mu)^2, \tag{2.2}$$

and

$$\zeta = \frac{t}{\gamma}, \qquad -1 < \zeta < 1. \tag{2.3}$$

The polynomials

$$P_{Nn}(x) \equiv \left(\frac{\gamma}{N}\right)^n P_n\left(\frac{Nx}{\gamma}\right) \tag{2.4}$$

are monic polynomials orthogonal with respect to the weight $e^{N\zeta x} m\left(\frac{Nx}{\gamma}\right)$, so that

$$\int_{-\infty}^{\infty} P_{Nn}(x) P_{Nm}(x) e^{N\zeta x} m\left(\frac{Nx}{\gamma}\right) dx = h_{Nn}\delta_{nm}, \tag{2.5}$$

where

$$h_{Nn} = \left(\frac{\gamma}{N}\right)^{2n+1} h_n. \tag{2.6}$$

It follows from (2.2) that

$$\tilde{\tau}_N = \prod_{n=0}^{N-1} h_{Nn}. \tag{2.7}$$

The polynomials $P_{Nn}(x)$ satisfy the three term recurrent relation,

$$xP_{Nn}(x) = P_{N,n+1}(x) + Q_{Nn}P_{Nn}(x) + R_{Nn}P_{N,n-1}(x), \tag{2.8}$$

where

$$R_{Nn} = \left(\frac{\gamma}{N}\right)^2 R_n, \qquad Q_{Nn} = \frac{\gamma}{N} Q_n. \tag{2.9}$$

In what follows we will evaluate the asymptotics of R_{NN} and Q_{NN} as $N \to \infty$. In particular, for R_{NN} we will obtain the formula

$$R_{NN} = R + \cos(N\omega) \sum_{j:\, \kappa_j \le 2} c_j N^{-\kappa_j} + O(N^{-2}). \tag{2.10}$$

Then (2.9) will provide us with the needed asymptotics of R_n as $n \to \infty$.

Define

$$V_N(\mu) = -\zeta\mu - \frac{1}{N} \ln \left[\frac{\sinh N\mu(\frac{\pi}{2\gamma} - 1)}{\sinh N\mu \frac{\pi}{2\gamma}} \right]. \tag{2.11}$$

Then

$$e^{-NV_N(\mu)} = e^{N\zeta\mu} \frac{\sinh N\mu(\frac{\pi}{2\gamma} - 1)}{\sinh N\mu \frac{\pi}{2\gamma}} = e^{N\zeta\mu} m\left(\frac{N\mu}{\gamma}\right), \tag{2.12}$$

hence

$$\tilde{\tau}_N = \int \prod_{i=1}^{N} [e^{-NV_N(\mu_i)} d\mu_i] \Delta(\mu)^2, \tag{2.13}$$

Observe that as $N \to \infty$,

$$V_N(\mu) \to V(\mu) \equiv -\zeta\mu + |\mu|. \tag{2.14}$$

We will evaluate the equilibrium measures, first for V and then for V_N. But before we discuss some general formulae for equilibrium measures.

3. Equilibrium Measure

In this section we remind some facts concerning equilibrium measures, see [8], [9]. Let $V(x)$ be a real analytic function such that

$$\lim_{x \to \pm\infty} \frac{V(x)}{\ln |x|} = \infty. \tag{3.1}$$

The equilibrium measure, $\nu^{\mathrm{eq}} = \nu_V^{\mathrm{eq}}$, for V is defined as a minimizer of the functional

$$I_V(\nu) = -\iint_{\mathbb{R}^2} \ln|x-y| d\nu(x) d\nu(y) + \int_{\mathbb{R}^1} V(x) d\nu(x), \tag{3.2}$$

over all probability measures ν on $\mathbb{R}^1$. The minimizer exists and it is unique. The equilibrium measure has the following properties:

- It is absolutely continuous with respect to the Lebesgue measure, $d\nu^{\mathrm{eq}}(x) = \rho(x)dx$.
- It is supported by a finite number of disjoint intervals, $S = \cup_{l=1}^{q} [\alpha_l, \beta_l]$.

- On S,

$$\rho(x) = \frac{1}{2\pi i} h(x)\sqrt{R(x+i0)}, \qquad R(z) \equiv \prod_{l=1}^{q} (z-\alpha_l)(z-\beta_l), \tag{3.3}$$

  where $h(x)$ is a real analytic function on the real line, and $\sqrt{R(z)}$ is taken on the principal sheet.

The function $h(x)$ is expressed by the contour integral,

$$h(x) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{V'(s)ds}{(x-s)\sqrt{R(s)}}, \qquad x \in S, \tag{3.4}$$

over any closed contour Γ, with S in its interior. For the equilibrium measure, consider its resolvent,

$$\omega(z) = \int_S \frac{\rho(\mu)d\mu}{z-\mu}, \quad z \in \mathbb{C} \setminus S, \tag{3.5}$$

and the g-function,

$$g(z) = \int_S \rho(\mu) \log(z-\mu) d\mu, \qquad z \in \mathbb{C} \setminus (-\infty, \beta_q], \tag{3.6}$$

where for $\log z$ the principal branch is taken. Then

$$g'(z) = \omega(z), \tag{3.7}$$

and, by the jump property of the Cauchy integral,

$$\omega(\mu+i0) - \omega(\mu-i0) = -2\pi i \rho(\mu), \qquad \mu \in S. \tag{3.8}$$

As $z \to \infty$,

$$\omega(z) = z^{-1} + O(z^{-2}), \qquad g(z) = \log z + O(z^{-1}). \tag{3.9}$$

The equilibrium measure is uniquely determined by the condition that there exists a real constant l such that

- For any $\mu \in S$,

$$g(\mu+i0) + g(\mu-i0) - V(\mu) - l = 0, \qquad \mu \in S. \tag{3.10}$$

- For any $\mu \in \mathbb{R} \setminus S$,

$$g(\mu+i0) + g(\mu-i0) - V(\mu) - l \le 0, \qquad \mu \in \mathbb{R} \setminus S, \tag{3.11}$$

see [8], [9]. It implies the equation,

$$\omega(\mu+i0) + \omega(\mu-i0) = V'(\mu), \qquad \mu \in S. \tag{3.12}$$

A solution to this equation can be found as

$$\omega(z) = \frac{\sqrt{R(z)}}{2\pi i} \int_S \frac{V'(x)dx}{(z-x)\sqrt{R(x+i0)}}. \tag{3.13}$$

From (3.8),

$$g(\mu+i0)-g(\mu-i0)=\begin{cases}2\pi i, & \mu\le\alpha_1,\\ 2\pi i\displaystyle\int_\mu^{\beta_q}\rho(s)\chi_S(s)ds, & \alpha_1\le\mu\le\beta_q,\\ 0, & \mu\ge\beta_q,\end{cases} \tag{3.14}$$

where χ_S is the characteristic function of S.

We will be interested in the case when V is convex. In this case the equilibrium measure is supported by one interval, say, $[\alpha,\beta]$, and (3.14) reduces to

$$g(\mu+i0)-g(\mu-i0)=\begin{cases}2\pi i, & \mu\le\alpha,\\ 2\pi i\displaystyle\int_\mu^{\beta}\rho(s)ds, & \alpha\le\mu\le\beta,\\ 0, & \mu\ge\beta.\end{cases} \tag{3.15}$$

For $z_0\in\mathbb{C}$ and $r>0$, we will us the standard notation for the disk,

$$D(z_0,r)=\{z\in\mathbb{C}:\ |z-z_0|<r\}. \tag{3.16}$$

From (3.3), (3.10) and (3.15), one obtains that there exists $r>0$ such that

$$-2g(z)+V(z)+l=\int_\beta^z h(s)\sqrt{(s-\alpha)(s-\beta)}ds,\qquad z\in D(\beta,r)\setminus[\alpha,\beta], \tag{3.17}$$

and

$$-2g(z)+V(z)+l=-2\pi i\,\mathrm{sgn}\,(\mathrm{Im}\,z)+\int_z^\alpha h(s)\sqrt{(\alpha-s)(\beta-s)}ds,\qquad z\in D(\alpha,r)\setminus\mathbb{R}. \tag{3.18}$$

Finally, it follows from (3.9) that the function $e^{Ng(z)}$ is analytic on $\mathbb{C}\setminus[\alpha,\beta]$, and

$$e^{Ng(z)}=z^N+O(z^{N-1}),\qquad z\to\infty. \tag{3.19}$$

4. Equilibrium Measure for V

In this section we consider the equilibrium measure for the potential

$$V(\mu)=-\zeta\mu+|\mu|,\qquad |\zeta|<1. \tag{4.1}$$

It is obviously a convex function, and

$$V'(\mu)=-\zeta+\mathrm{sgn}\,(\mu). \tag{4.2}$$

Integral (3.13) is explicitly evaluated in this case as

$$\omega(z)=\frac{1-\zeta}{2}+\frac{2}{i\pi}\log\left[\frac{\sqrt{\beta(z-\alpha)}-i\sqrt{-\alpha(z-\beta)}}{\sqrt{z(\beta-\alpha)}}\right], \tag{4.3}$$

and from the asymptotics,

$$\omega(z)=z^{-1}+O(z^{-2}),\qquad z\to\infty, \tag{4.4}$$

one finds that

$$\alpha=-\pi\tan\frac{\pi}{4}(1-\zeta),\quad \beta=\pi\tan\frac{\pi}{4}(1+\zeta), \tag{4.5}$$

see [28]. Observe that

$$\beta+\alpha=2\pi\tan\frac{\pi\zeta}{2}\,,\qquad \beta-\alpha=\frac{2\pi}{\cos\frac{\pi\zeta}{2}}\,,\qquad (-\alpha)\beta=\pi^2. \tag{4.6}$$

For the square root in (4.3) we take the principal branch, with a cut on the negative half-axis. The function $\omega(z)$ is analytic on $\mathbb{C}\setminus[\alpha,\beta]$. On $[\alpha,\beta]$,

$$\omega(\mu\pm i0)=\frac{-\zeta+\operatorname{sgn}(\mu)}{2}\pm\frac{2}{i\pi}\log\left[\frac{\sqrt{\beta(\mu-\alpha)}+\sqrt{-\alpha(\beta-\mu)}}{\sqrt{|\mu|(\beta-\alpha)}}\right],\qquad \alpha<\mu<\beta, \tag{4.7}$$

so that the density function $\rho(\mu)$ is equal to

$$\rho(\mu)=\frac{2}{\pi^2}\log\left[\frac{\sqrt{\beta(\mu-\alpha)}+\sqrt{-\alpha(\beta-\mu)}}{\sqrt{|\mu|(\beta-\alpha)}}\right],\qquad \alpha<\mu<\beta. \tag{4.8}$$

Observe that $\rho(\mu)$ has a logarithmic singularity at the origin. From (4.7),

$$\omega(\alpha)=\frac{-\zeta-1}{2}\,,\quad \omega(\beta)=\frac{-\zeta+1}{2}\,. \tag{4.9}$$

From (4.3), (4.5) we obtain that

$$g(z)=z\omega(z)+2\log\left(\sqrt{z-\alpha}+\sqrt{z-\beta}\right)-(1+2\log 2) \tag{4.10}$$

(see Appendix B below). This implies that

$$\int_\mu^\beta \rho(s)ds=-\mu\rho(\mu)+\frac{2}{\pi}\arctan\sqrt{\frac{\beta-\mu}{\mu-\alpha}},\qquad \alpha\le\mu\le\beta. \tag{4.11}$$

In particular,

$$\int_0^\beta \rho(s)ds=\frac{1+\zeta}{2}\,. \tag{4.12}$$

In addition, we have that

$$g(\mu+i0)+g(\mu-i0)-V(\mu)-l=0,\quad \alpha\le\mu\le\beta. \tag{4.13}$$

By (4.9), $\omega(\beta)=\frac{1-\zeta}{2}$, hence

$$l=2g(\beta)-V(\beta)=2\ln(\beta-\alpha)-2-4\ln 2. \tag{4.14}$$

Define an analytic continuation of the potential $V(\mu)=-\zeta\mu+|\mu|$ from $\mathbb{R}$ to $\mathbb{C}$ as

$$V(z)=\begin{cases}(1-\zeta)z, & \operatorname{Re}z\ge 0,\\ (-1-\zeta)z, & \operatorname{Re}z\le 0\end{cases} \tag{4.15}$$

(the function $V(z)$ is two-valued on $\operatorname{Re}z=0$). In what follows we will use the function $h(z)$ defined by the formulae,

$$h(z)=\frac{4i}{\pi\sqrt{(z-\alpha)(z-\beta)}}\log\left[\frac{\sqrt{\beta(z-\alpha)}-i\sqrt{(-\alpha)(z-\beta)}}{\sqrt{z(\beta-\alpha)}}\right],\quad \operatorname{Re}z\ge 0. \tag{4.16}$$

and

$$h(z) = -\frac{4i}{\pi\sqrt{(\alpha-z)(\beta-z)}}\log\left[\frac{\sqrt{(-\alpha)(\beta-z)}+i\sqrt{\beta(\alpha-z)}}{\sqrt{(-z)(\beta-\alpha)}}\right], \quad \operatorname{Re} z \le 0, \tag{4.17}$$

where the square root and logarithm are taken on the principal sheet. The function $h(z)$ has the following properties:

(i) $h(z)$ is analytic in $\mathbb{C}\setminus i\mathbb{R}$ and

$$h(\alpha) = \frac{4}{(-\alpha)(\beta-\alpha)} > 0, \quad h(\beta) = \frac{4}{\beta(\beta-\alpha)} > 0; \tag{4.18}$$

$$h'(\alpha) = \frac{4(\beta-3\alpha)}{3\alpha^2(\beta-\alpha)^2}, \quad h'(\beta) = \frac{4(\alpha-3\beta)}{3\beta^2(\beta-\alpha)^2}; \tag{4.19}$$

(ii) by (4.8),

$$\rho(\mu) = \frac{1}{2\pi}h(\mu)\sqrt{(\mu-\alpha)(\beta-\mu)}, \quad \alpha<\mu<\beta; \tag{4.20}$$

(iii) by (4.3),

$$\omega(z) = \frac{V'(z)}{2} - \frac{h(z)\sqrt{(z-\alpha)(z-\beta)}}{2} \tag{4.21}$$

Asymptotic formulae for orthogonal polynomials with weight (4.1) with $\zeta = 0$ were obtained, via the Riemann-Hilbert approach, in paper [15] by Kriecherbauer and McLaughlin. In fact, they studied a more general case, of the Freud potentials of the form $V(\mu) = |\mu|^\alpha$.

5. Equilibrium Measure for V_N

From (2.11) we obtain that

$$V_N'(\mu) = -\left(\frac{\pi}{2\gamma}-1\right)\coth N\mu\left(\frac{\pi}{2\gamma}-1\right) + \frac{\pi}{2\gamma}\coth N\mu\frac{\pi}{2\gamma} - \zeta. \tag{5.1}$$

The function $V_N'(\mu)$ is increasing, hence $V_N(\mu)$ is convex. Its equilibrium measure, $\rho_N(\mu)d\mu$, is supported by one interval $[\alpha_N, \beta_N]$. As $N\to\infty$, the equilibrium measure for V_N converges to the one for V. In this section we will derive some asymptotic formulas for α_N, β_N and $\rho_N(\mu)$ as $N\to\infty$. Consider the resolvent,

$$\omega_N(z) = \int_{\alpha_N}^{\beta_N}\frac{d\mu\,\rho_N(\mu)}{z-\mu}, \quad z\in\mathbb{C}\setminus[\alpha_N,\beta_N]. \tag{5.2}$$

Then

$$\rho_N(\mu) = -\frac{1}{2\pi i}[\omega_N(\mu+i0)-\omega_N(\mu-i0)], \qquad \alpha_N<\mu<\beta_N, \tag{5.3}$$

and

$$\omega_N(\mu\pm i0) = \mp\pi i\rho_N(\mu) + P.V.\int_{\alpha_N}^{\beta_N}\frac{\rho_N(x)dx}{\mu-x}, \qquad \alpha_N<\mu<\beta_N, \tag{5.4}$$

where $P.V.\int$ is the principal value of the integral. The resolvent solves the equation,

$$\omega_N(\mu+i0)+\omega_N(\mu-i0) = V_N'(\mu), \qquad \alpha_N<\mu<\beta_N. \tag{5.5}$$

The solution to this equation is

$$\omega_N(z) = -\frac{\sqrt{R_N(z)}}{2\pi i} \int_{\alpha_N}^{\beta_N} \frac{V_N'(x)}{(z-x)\sqrt{R_N(x)}_+} dx, \quad z \in \mathbb{C} \setminus [\alpha_N, \beta_N], \tag{5.6}$$

where

$$R_N(z) = (z-\alpha_N)(z-\beta_N), \tag{5.7}$$

and $\sqrt{R_N(z)}$ is taken on the principal sheet, with a cut on $[\alpha_N, \beta_N]$. As usual,

$$\sqrt{R_N(x)}_+ = \lim_{\varepsilon \to +0} \sqrt{R_N(x+i\varepsilon)}.$$

Evaluation of the end-points. From (5.2) we have that $\omega_N(z) \sim \frac{1}{z}$ as $z \to \infty$. By evaluating the large z asymptotics of the integral on the right in (5.6), we obtain the equations,

$$\frac{1}{2\pi} \int_{\alpha_N}^{\beta_N} \frac{V_N'(x)}{\sqrt{(x-\alpha_N)(\beta_N - x)}} dx = 0, \tag{5.8}$$

and

$$\frac{1}{2\pi} \int_{\alpha_N}^{\beta_N} \frac{x V_N'(x)}{\sqrt{(x-\alpha_N)(\beta_N - x)}} dx = 1. \tag{5.9}$$

From these two equations we obtain the following asymptotics of α_N, β_N as $N \to \infty$.

Proposition 5.1. *As $N \to \infty$,*

$$\begin{aligned} \alpha_N &= \alpha + N^{-2} \frac{\gamma^2 \left(2 \sin \frac{\pi\zeta}{2} - 1\right)}{3(\pi - 2\gamma)\cos\frac{\pi\zeta}{2}} + O(N^{-3}), \\ \beta_N &= \beta + N^{-2} \frac{\gamma^2 \left(2 \sin \frac{\pi\zeta}{2} + 1\right)}{3(\pi - 2\gamma)\cos\frac{\pi\zeta}{2}} + O(N^{-3}), \end{aligned} \tag{5.10}$$

where α, β are given in (4.5).

Proof of Proposition 5.1 is given in Appendix C below.

Evaluation of the density. Consider now the asymptotics of the density function $\rho_N(x)$. As $N \to \infty$, it approaches the density function $\rho(x)$ given in (4.8). The density $\rho(x)$ has a logarithmic singularity at $x = 0$. For ρ_N the singularity is smoothed out and we are interested in the large N asymptotics of ρ_N near the origin. From (5.3) and (5.6) we obtain that

$$\rho_N(\mu) = -\frac{\sqrt{r_N(\mu)}}{2\pi^2} P.V. \int_{\alpha_N}^{\beta_N} \frac{V_N'(x)dx}{(\mu - x)\sqrt{r_N(x)}}, \qquad \alpha_N < \mu < \beta_N. \tag{5.11}$$

where

$$r_N(x) = (x-\alpha_N)(\beta_N - x). \tag{5.12}$$

Observe that $\rho_N(\mu)$ is analytic for $\alpha_N < \mu < \beta_N$. From (5.1) we have that

$$V_N'(x) = \operatorname{sgn} x - \zeta + f(Nx), \tag{5.13}$$

where

$$f(x) = \frac{\pi}{2\gamma} \coth x \frac{\pi}{2\gamma} - \left(\frac{\pi}{2\gamma} - 1\right) \coth x \left(\frac{\pi}{2\gamma} - 1\right) - \operatorname{sgn} x, \tag{5.14}$$

hence

$$\rho_N(\mu) = \rho_N^0(\mu) + \rho_N^1(\mu), \tag{5.15}$$

where

$$\rho_N^0(\mu) = -\frac{\sqrt{r_N(\mu)}}{2\pi^2} P.V. \int_{\alpha_N}^{\beta_N} \frac{(\operatorname{sgn} x - \zeta)dx}{(\mu - x)\sqrt{r_N(x)}}, \qquad \alpha_N < \mu < \beta_N, \tag{5.16}$$

and

$$\rho_N^1(\mu) = -\frac{\sqrt{r_N(\mu)}}{2\pi^2} P.V. \int_{\alpha_N}^{\beta_N} \frac{f(Nx)dx}{(\mu - x)\sqrt{r_N(x)}}, \qquad \alpha_N < \mu < \beta_N. \tag{5.17}$$

The function $\rho_N^0(\mu)$ is evaluated explicitly as

$$\rho_N^0(\mu) = \frac{2}{\pi^2} \log \left[\frac{\sqrt{\beta_N(\mu - \alpha_N)} + \sqrt{-\alpha_N(\beta_N - \mu)}}{\sqrt{|\mu|(\beta_N - \alpha_N)}} \right], \quad \alpha_N < \mu < \beta_N. \tag{5.18}$$

[cf. (4.8)]. Set

$$\omega_N^0(z) = \int_{\alpha_N}^{\beta_N} \frac{d\mu\, \rho_N^0(\mu)}{z - \mu}, \quad z \in \mathbb{C} \setminus [\alpha_N, \beta_N]. \tag{5.19}$$

Then

$$\omega_N^0(z) = \frac{1-\zeta}{2} + \frac{2}{i\pi} \log \left[\frac{\sqrt{\beta_N(z - \alpha_N)} - i\sqrt{-\alpha_N(z - \beta_N)}}{\sqrt{z(\beta_N - \alpha_N)}} \right], \tag{5.20}$$

[cf. (4.3)].

To describe the large N asymptotics of $\rho_N^1(\mu)$, introduce the function,

$$k(\mu) = P.V. \int_{-\infty}^{\infty} \frac{f(x)dx}{\mu - x}. \tag{5.21}$$

From expilicit formula (5.14), we have the following properties of the function $f(x)$:

- $f(x)$ satisfies the estimate,
$$|f(x)| \le C_0 e^{-C|x|}, \tag{5.22}$$
with some $C_0, C > 0$;
- $f(x)$ is an odd function;
- the function $f(x) + \operatorname{sgn} x$ is real analytic.

These properties of $f(x)$ imply the following properties of the function $k(\mu)$:

- the function $k(\mu)$ is even and
$$k(\mu) = -2\ln|\mu| + k_0(\mu), \tag{5.23}$$
where $k_0(\mu)$ is real analytic;
- as $\mu \to \infty$,
$$k(\mu) = \frac{C}{\mu^2} + O(\mu^{-4}), \qquad \mu \to \infty, \tag{5.24}$$
where
$$C = \int_{-\infty}^{\infty} x f(x)dx = -\frac{2\pi\gamma^2}{3(\pi - 2\gamma)}. \tag{5.25}$$

We use the properties of $k(\mu)$ to prove the following asymptotics of the function $\rho_N^1(\mu)$.

Proposition 5.2. *As $N \to \infty$, uniformly in the interval $\alpha_N \le \mu \le \beta_N$,*

$$\rho_N^1(\mu) = -\frac{1}{2\pi^2}\, k(N\mu) + O(N^{-2}). \tag{5.26}$$

In addition, there exists a family of complex domains $\{\Omega_r,\ r > 0\}$ such that $[\alpha_N + r, \beta_N - r] \subset \Omega_r$ and $\Omega_r \subset \Omega_{r'}$ whenever $r > r'$, and such that the function

$$\varepsilon_N(\mu) \equiv \rho_N^1(\mu) + \frac{1}{2\pi^2}\, k(N\mu). \tag{5.27}$$

can be analytically continued to Ω_r and as $N \to \infty$,

$$\sup_{z\in\Omega_r} |\varepsilon_N(z)| = O(N^{-2}). \tag{5.28}$$

The proof of Proposition 5.2 is given below in Appendix D.

Proposition 5.2 implies that

$$\rho_N(\mu) = \rho_N^0(\mu) - \frac{1}{2\pi^2}\, k(N\mu) + O(N^{-2}), \qquad \mu \in [\alpha_N, \beta_N], \tag{5.29}$$

and this equation can be extended to the complex domain Ω_r, $r > 0$. This can be further specified as follows. Let $r > 0$ be an arbitrary fixed number such that $r \le \frac{1}{2}\min\{-\alpha, \beta\}$. Then

- For $\mu \in [\alpha_N + r, \beta_N - r]$,
$$\rho_N(\mu) = \rho(\mu) - \frac{1}{2\pi^2}\, k(N\mu) + O(N^{-2}), \tag{5.30}$$
where $\rho(\mu)$ is given in (4.8).
- For $\mu \in [\alpha_N, \alpha_N + r] \cup [\beta_N - r, \beta_N]$,
$$\rho_N(\mu) = \rho_N^0(\mu) + O(N^{-2}). \tag{5.31}$$

Observe that (5.30) implies that

$$\rho_N(\mu) = \frac{1}{\pi^2}\ln N + a(\mu) - \frac{1}{2\pi^2}\, k_0(N\mu) + O(N^{-2}), \qquad \mu \in [\alpha_N + r, \beta_N - r], \tag{5.32}$$

where

$$a(\mu) = \frac{2}{\pi^2}\log\left[\frac{\sqrt{\beta(\mu-\alpha)} + \sqrt{-\alpha(\beta-\mu)}}{\sqrt{\beta-\alpha}}\right], \quad \alpha < \mu < \beta. \tag{5.33}$$

From (5.29) we obtain the following result:

Proposition 5.3. *We have that*

$$\int_0^{\beta_N} \rho_N(\mu) d\mu = \frac{1+\zeta}{2} + O(N^{-2}). \tag{5.34}$$

Proof. By an explicit integration of (5.18) we have that

$$\int_\mu^{\beta_N} \rho_N^0(x) dx = -\mu\rho_N^0(\mu) + \frac{2}{\pi}\arctan\sqrt{\frac{\beta_N - \mu}{\mu - \alpha_N}}, \qquad \alpha_N \le \mu \le \beta_N. \tag{5.35}$$

In particular,

$$\int_{\alpha_N}^{\beta_N} \rho_N^0(x) dx = 1, \tag{5.36}$$

and

$$\int_0^{\beta_N} \rho_N^0(x)dx = \frac{1+\zeta}{2} + O(N^{-2}). \tag{5.37}$$

Since

$$\int_{\alpha_N}^{\beta_N} \rho_N(x)dx = 1, \tag{5.38}$$

we obtain from (5.29) that

$$\int_{-\infty}^{\infty} k(x)dx = 0. \tag{5.39}$$

Since $k(x)$ is even, this implies that

$$\int_0^{\infty} k(x)dx = 0. \tag{5.40}$$

Therefore,

$$\int_0^{\beta_N} \rho_N(\mu)d\mu = \int_0^{\beta_N} \rho_N^0(\mu)d\mu - \frac{1}{2\pi^2}\int_0^{\beta_N} k(N\mu)d\mu + O(N^{-2}) = \frac{1+\zeta}{2} + O(N^{-2}). \tag{5.41}$$

Proposition 5.3 is proved.

Evaluation of the resolvent. The large N asymptotics of the function $\omega_N(z)$ can be obtained as follows. By (5.15) and (5.27),

$$\rho_N(\mu) = \rho_N^0(\mu) - \frac{1}{2\pi^2}\, k(N\mu) + \varepsilon_N(\mu), \tag{5.42}$$

hence

$$\omega_N(z) = \omega_N^0(z) - \frac{1}{2\pi^2}\, m(Nz) + \xi_N(z), \tag{5.43}$$

where

$$m(z) = \int_{-\infty}^{\infty} \frac{k(\mu)d\mu}{z-\mu}, \qquad \xi_N(z) = \int_{-\infty}^{\infty} \frac{\varepsilon_N(\mu)d\mu}{z-\mu}. \tag{5.44}$$

Observe that $\xi_N(z)$ is an analytic function in $(\Omega_r \setminus \mathbb{R})$, $r > 0$. Consider a complex domain U_r such that the closure of U_r belongs to Ω_r and $[\alpha_N + r, \beta_N - r] \subset U_r$. Then from (5.28) and the analyticity of $\varepsilon_N(z)$ in Ω_r we obtain that

$$\sup_{z\in U_r\setminus\mathbb{R}} |\xi_N(z)| = O(N^{-2}). \tag{5.45}$$

We have that

$$m(z) = \operatorname{sgn}(\operatorname{Im} z)\pi i \int_{-\infty}^{\infty} \frac{f(\mu)d\mu}{z-\mu}, \tag{5.46}$$

Indeed, if we introduce the Fourier transform,

$$\tilde{k}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\tau\mu} k(\mu)d\mu, \tag{5.47}$$

then (5.44) implies that

$$\tilde{m}_\pm(\tau) = \mp 2\pi i\theta(\pm\tau)\tilde{k}(\tau), \tag{5.48}$$

where $m_\pm(\mu) = m(\mu \pm i0)$, and $\theta(\tau) = 1$ for $\tau \ge 0$ and $\theta(\tau) = 0$ for $\tau < 0$. Also, from (5.21),

$$\tilde{k}(\tau) = -\pi i \operatorname{sgn}(\tau)\tilde{f}(\tau), \tag{5.49}$$

hence

$$\tilde{m}_{\pm}(\tau) = -2\pi^2\theta(\pm\tau)\tilde{f}(\tau). \tag{5.50}$$

By taking the inverse Fourier transforms, we obtain that

$$m(\mu \pm i0) = \pm\pi i \int_{-\infty}^{\infty} \frac{f(x)dx}{\mu \pm i0 - x}\,, \tag{5.51}$$

which implies, by means of analytic continuation, (5.46).

By using the listed above properties of $f(x)$, we obtain from (5.46) the following properties of $m(z)$, $z \in (\mathbb{C} \setminus \mathbb{R})$:

- The symmetry conditions,

$$m(-z) = -m(z), \qquad m(\bar{z}) = \overline{m(z)}. \tag{5.52}$$

- The representation,

$$m(z) = \pm 2\pi i \log z + m_0(z), \qquad \pm \mathrm{Im}\, z > 0, \tag{5.53}$$

  with $\log z$ on the principal sheet, where $m_0(z)$ is analytic in the closed half-planes $\{\pm \mathrm{Im}\, z \geq 0\}$, and

$$m_0(-z) = -m_0(z), \qquad m_0(\bar{z}) = \overline{m_0(z)}. \tag{5.54}$$

- As $z \to \infty$,

$$m(z) = \frac{C \mathrm{sgn}\,(\mathrm{Im}\, z)}{z^2} + O(z^{-4}). \tag{5.55}$$

We summarize the properties of $\omega_N(z)$ in the following proposition.

Proposition 5.4. *For any $r > 0$ there exists an independent of N complex neighborhood U_r of the interval $\alpha_N + r \leq \mu \leq \beta_N - r$ such that for $z \in U_r$, equation (5.43) holds, in which $\omega_N^0(z)$ is given by (5.20), $m(z)$, by (5.46), and $\xi_N(z)$ satisfies estimate (5.45). In addition, for $z \in U_r$,*

$$\omega_N(z) = \mp \frac{i \ln N}{\pi} + b(z) - \frac{1}{2\pi^2} m_0(Nz) + O(N^{-2}), \qquad \pm \mathrm{Im}\, z > 0, \tag{5.56}$$

where

$$b(z) = \frac{1-\zeta}{2} + \frac{2}{i\pi} \log \left[\frac{\sqrt{\beta(z-\alpha)} - i\sqrt{-\alpha(z-\beta)}}{\sqrt{\beta - \alpha}} \right], \tag{5.57}$$

with a cut on $[\alpha, \beta]$, and

$$m_0(z) = \pm\pi i \left[\int_{-\infty}^{\infty} \frac{f(\mu)d\mu}{z-\mu} - 2\log z \right], \qquad \pm \mathrm{Im}\, z > 0, \tag{5.58}$$

with $\log z$ on the principal sheet.

Observe that both $b(z)$ and $m_0(Nz)$ have a jump across $[\alpha, \beta]$, and

$$b(\bar{z}) = \overline{b(z)}, \qquad m_0(\bar{z}) = \overline{m_0(z)}. \tag{5.59}$$

By using (4.6), we find that

$$b(+i0) = \frac{1-\zeta}{2} + \frac{1}{i\pi} \ln \left(2\pi \cos \frac{\pi\zeta}{2} \right). \tag{5.60}$$

Evaluation of the constant of integration. Let us evaluate l_N. By (3.10), for any $\mu \in [\alpha_N, \beta_N]$,

$$l_N = V_N(\mu) - g_{N-}(\mu) - g_{N+}(\mu). \tag{5.61}$$

Take $\mu = \frac{\beta}{2}$. By (2.11), there exists $c > 0$ such that

$$V_N\left(\frac{\beta}{2}\right) = V\left(\frac{\beta}{2}\right) + O(e^{-cN})\,. \tag{5.62}$$

Also, by (3.6),

$$g_{N-}\left(\frac{\beta}{2}\right) + g_{N+}\left(\frac{\beta}{2}\right) = 2\int_{\alpha_N}^{\beta_N} \rho_N(x)\ln\left|\frac{\beta}{2} - x\right|\,dx \tag{5.63}$$

By (5.29) and (5.10), we can reduce this to

$$\begin{aligned} g_{N-}\left(\frac{\beta}{2}\right) + g_{N+}\left(\frac{\beta}{2}\right) &= 2\int_{\alpha}^{\beta} \rho(x)\ln\left|\frac{\beta}{2} - x\right|\,dx \\ &\quad - \frac{1}{\pi^2}\int_{\alpha_N}^{\beta_N} k(Nx)\ln\left|\frac{\beta}{2} - x\right|\,dx + O(N^{-2}). \end{aligned} \tag{5.64}$$

From (5.24) and (5.39) we obtain that

$$\int_{\alpha_N}^{\beta_N} k(Nx)\ln\left|\frac{\beta}{2} - x\right|\,dx = O(N^{-2}), \tag{5.65}$$

hence

$$g_{N-}\left(\frac{\beta}{2}\right) + g_{N+}\left(\frac{\beta}{2}\right) = 2\int_{\alpha}^{\beta} \rho(x)\ln\left|\frac{\beta}{2} - x\right|\,dx + O(N^{-2}). \tag{5.66}$$

Thus,

$$\begin{aligned} l_N &= V_N\left(\frac{\beta}{2}\right) - g_{N-}\left(\frac{\beta}{2}\right) - g_{N+}\left(\frac{\beta}{2}\right) \\ &= V\left(\frac{\beta}{2}\right) - g_{-}\left(\frac{\beta}{2}\right) - g_{+}\left(\frac{\beta}{2}\right) + O(N^{-2}) = l + O(N^{-2}), \end{aligned} \tag{5.67}$$

where by (4.14), $l = 2\ln(\beta - \alpha) - 2 - 4\ln 2$.

6. Riemann-Hilbert Problem

The Riemann-Hilbert (RH) problem for orthogonal polynomials with respect to the weight $w(\mu)$ is the following:

(i) *(analyticity)* $Y(z) = (Y_{ij}(z))_{i,j=1,2}$ is a matrix valued analytic function on $\mathbb{C}\setminus\mathbb{R}$ which has limits on the real line, $Y_\pm(\mu)$, so that for all $A > 0$,

$$\lim_{\varepsilon\to 0^+}\max_{-A\le\mu\le A}|Y(\mu \pm i\varepsilon) - Y_\pm(\mu)| = 0. \tag{6.1}$$

(ii) *(jump condition)*

$$Y_+(\mu) = Y_-(\mu)\begin{pmatrix} 1 & w(\mu) \\ 0 & 1 \end{pmatrix}. \tag{6.2}$$

(iii) *(asymptotics at infinity)*

$$Y(z) = \left[I + O\left(|z|^{-1}\right)\right]\begin{pmatrix} z^n & 0 \\ 0 & z^{-n}\end{pmatrix}, \quad |z| \to \infty. \tag{6.3}$$

Proposition 6.1. *The RH problem (i)-(iii) has a unique solution given by*

$$Y(z) = \begin{pmatrix} \pi_n(z) & \int_{\mathbb{R}} \frac{\pi_n(\mu)w(\mu)d\mu}{(\mu-z)2\pi i} \\ -\frac{2\pi i \pi_{n-1}(z)}{h_{n-1}} & \int_{\mathbb{R}} \frac{-\pi_{n-1}(\mu)w(\mu)d\mu}{(\mu-z)h_{n-1}} \end{pmatrix} \tag{6.4}$$

where $\pi_n(\mu) = \mu^n + \dots$ denotes the n-th monic orthogonal polynomial with respect to the measure $w(\mu)d\mu$ and $h_n = \int_{\mathbb{R}} \pi_n(\mu)^2 w(\mu)d\mu$. Furthermore, there exist 2×2 matrices $Y_j,\ j = 1,2,\dots,$ so that for all $m \ge 1$,

$$Y(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n\end{pmatrix} = I + \frac{Y_1}{z} + \frac{Y_2}{z^2} + \dots + \frac{Y_m}{z^m} + O\left(|z|^{-m-1}\right), \quad |z| \to \infty, \tag{6.5}$$

and

$$\begin{aligned} h_n &= -2\pi i (Y_1)_{12}, \quad h_{n-1} = -\frac{2\pi i}{(Y_1)_{21}} \\ R_n &= (Y_1)_{21}(Y_1)_{12}, \\ Q_n &= \frac{(Y_2)_{21}}{(Y_1)_{21}} + (Y_1)_{11}, \end{aligned} \tag{6.6}$$

where $Q_n,\ R_n$ are the recurrence coefficients associated to the orthogonal polynomials,

$$z\pi_n(z) = \pi_{n+1}(z) + Q_n\pi_n(z) + R_n\pi_{n-1}(z). \tag{6.7}$$

RH problem (i)-(iii) and Proposition 1 hold for a general weight $w(\mu)$ (see [2] for conditions on $w(\mu)$). In our case

$$w(\mu) = e^{-NV_N(\mu)}, \tag{6.8}$$

and (6.6) reads

$$\begin{aligned} h_{Nn} &= -2\pi i (Y_1)_{12}, \quad h_{N,n-1} = -\frac{2\pi i}{(Y_1)_{21}} \\ R_{Nn} &= (Y_1)_{21}(Y_1)_{12}, \\ Q_{Nn} &= \frac{(Y_2)_{21}}{(Y_1)_{21}} + (Y_1)_{11}, \end{aligned} \tag{6.9}$$

7. Transformations of the RH Problem

We will follow [9] to find the asymptotics of the solution $Y(z)$ to the Riemann-Hilbert problem (i)-(iii) in the case when $n = N$ and $N \to \infty$.

Transformation of the RH problem (6.1)-(6.3). Set

$$T(z) \equiv e^{-N\frac{l_N}{2}\sigma_3} Y(z) e^{-N\left(g_N(z) - \frac{l_N}{2}\right)\sigma_3}, \quad z \in \mathbb{C}\setminus\mathbb{R}; \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}. \tag{7.1}$$

where

$$g_N(z) = \int_{\alpha_N}^{\beta_N} \rho_N(\mu)\log(z-\mu)d\mu. \tag{7.2}$$

Then $T(z)$ solves the following RH problem:

(i) *(analyticity)* $T(z) = (T_{ij}(z))_{i,j=1,2}$ is a matrix valued analytic function on $\mathbb{C}\setminus\mathbb{R}$ which has limits on the real line, $T_\pm(\mu)$, so that for all $A > 0$,

$$\lim_{\varepsilon\to 0^+} \max_{-A\le\mu\le A} |T(\mu \pm i\varepsilon) - T_\pm(\mu)| = 0. \tag{7.3}$$

(ii) *(jump condition)*

$$T_+(\mu) = T_-(\mu) J_T(\mu), \tag{7.4}$$

where

$$J_T(\mu) = \begin{pmatrix} e^{-N(g_{N+}(\mu)-g_{N-}(\mu))} & 1 \\ 0 & e^{N(g_{N+}(\mu)-g_{N-}(\mu))} \end{pmatrix}, \quad \mu \in [\alpha_N, \beta_N], \tag{7.5}$$

and

$$J_T(\mu) = \begin{pmatrix} 1 & e^{N[g_{N+}(\mu)+g_{N-}(\mu)-V_N(\mu)-l_N]} \\ 0 & 1 \end{pmatrix}, \quad \mu \in \mathbb{R}\setminus[\alpha_N, \beta_N]. \tag{7.6}$$

(iii) *(asymptotics at infinity)*

$$T(z) = I + O\left(|z|^{-1}\right), \quad |z| \to \infty. \tag{7.7}$$

The key point here is that the (21) element of the matrix $J_T(\mu)$ on $[\alpha_N, \beta_N]$ is equal to 1, due to equation (3.10). For convenience, let us rewrite the recurrent coefficients Q_{NN}, R_{NN} in the new terms:

$$\begin{aligned} &h_{NN} = -2\pi i e^{Nl_N}(T_1)_{12}, \quad h_{N,N-1} = -\frac{2\pi i e^{Nl_N}}{(T_1)_{21}} \\ &R_{NN} = (T_1)_{21}(T_1)_{12}, \\ &Q_{NN} = \frac{(T_2)_{21}}{(T_1)_{21}} + (T_1)_{11}. \end{aligned} \tag{7.8}$$

Jump matrix factorization. Denote for the sake of brevity

$$G_N(\mu) = g_{N+}(\mu) - g_{N-}(\mu). \tag{7.9}$$

There is the following factorization of the jump matrix J_T on $[\alpha_N, \beta_N]$:

$$\begin{pmatrix} e^{-NG_N(\mu)} & 1 \\ 0 & e^{NG_N(\mu)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{NG_N(\mu)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-NG_N(\mu)} & 1 \end{pmatrix} \equiv v_- v_0 v_+. \tag{7.10}$$

Substituting this factorization into (7.4) for $\mu \in [\alpha_N, \beta_N]$, we obtain that

$$T_+(\mu) = T_-(\mu) v_-(\mu) v_0 v_+(\mu), \tag{7.11}$$

or

$$\left[T_+(\mu) v_+^{-1}(\mu)\right] = \left[T_-(\mu) v_-(\mu)\right] v_0, \quad \mu \in [\alpha_N, \beta_N]. \tag{7.12}$$

Lenses. By using the factorization of jump matrix (7.10) above, we can transform the RH problem for T in the following way. Consider the contours Σ_N^+ and Σ_N^- on the complex plane from α_N to β_N, as shown on Figure 6.

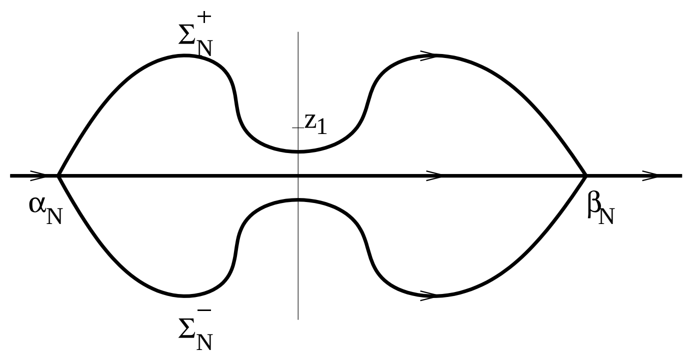

FIGURE 6. The lenses.

The contours $\Sigma_N^{\pm}$ go closer and closer to the origin as $N \to \infty$. Namely, by (2.12),

$$e^{NV_N(z)} = e^{-N\zeta z} \frac{\sinh Nz\frac{\pi}{2\gamma}}{\sinh Nz(\frac{\pi}{2\gamma}-1)}, \tag{7.13}$$

so that the function $e^{NV_N(z)}$ has poles on the imaginary axis. Consider the first pole in the upper half-plane,

$$z_1 = \frac{iN^{-1}\pi}{\frac{\pi}{2\gamma}-1}. \tag{7.14}$$

The contour Σ_N^+ should be in the upper half-plane and it should cross the imaginary axis below z_1, say, at $\frac{1}{2}z_1$. We take $\Sigma_N^- = \overline{\Sigma_N^+}$. We call the region between Σ_N^+ (respectively, Σ_N^-) and $[\alpha_N, \beta_N]$ the upper (respectively, lower) lens. Let

$$S(z) = \begin{cases} T(z), & \text{outside of the lenses,} \\ T(z)[v_+(z)]^{-1}, & \text{in the upper lens,} \\ T(z)v_-(z), & \text{in the lower lens.} \end{cases} \tag{7.15}$$

Then $S(z)$ solves the following RH problem:

(i) *(analyticity)* $S(z)$ is analytic on $\mathbb{C} \setminus (\mathbb{R} \cup \Sigma)$.

(ii) *(jump condition)*

$$S_+(z) = S_-(z)J_S(z), \tag{7.16}$$

where

$$J_S(z) = \begin{cases} \begin{pmatrix} 1 & e^{N[g_{N+}(z)+g_{N-}(z)-V_N(z)-l_N]} \\ 0 & 1 \end{pmatrix}, & z \in \mathbb{R} \setminus [\alpha_N, \beta_N], \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in [\alpha_N, \beta_N], \\ \begin{pmatrix} 1 & 0 \\ e^{-NG_N(z)} & 1 \end{pmatrix}, & z \in \Sigma_N^+, \\ \begin{pmatrix} 1 & 0 \\ e^{NG_N(z)} & 1 \end{pmatrix}, & z \in \Sigma_N^-. \end{cases} \tag{7.17}$$

(iii) *(asymptotics at infinity)*

$$S(z) = I + O\left(|z|^{-1}\right), \quad |z| \to \infty. \tag{7.18}$$

Evaluation of the functions $e^{\pm NG_N(z)}$. By (7.9) and (3.15),

$$G_N(\mu) = 2\pi i \int_{\mu}^{\beta_N} \rho_N(s)ds, \qquad \alpha_N \le \mu \le \beta_N, \tag{7.19}$$

hence, in particular, by (5.34),

$$G_N(0) = 2\pi i \int_0^{\beta_N} \rho_N(s)ds = \pi i(1+\zeta) + O(N^{-2}). \tag{7.20}$$

Consider first $e^{-NG_N(z)}$ on Σ_N^+. From (3.14) we have that

$$g_{N-}(\mu) = V_N(\mu) + l_N - g_{N+}(\mu), \qquad \alpha_N \le \mu \le \beta_N. \tag{7.21}$$

The RHS of this equation is extended to $\operatorname{Im}\mu > 0$ and this gives us an analytic continuation of $g_{N-}(\mu)$. By applying this continuation to (7.9), we obtain that for $z \in \Sigma_N^+$,

$$G_N(z) = 2g_N(z) - V_N(z) - l_N, \tag{7.22}$$

hence

$$e^{-NG_N(z)} = e^{-2Ng_N(z) + NV_N(z) + Nl_N}. \tag{7.23}$$

By (2.12),

$$e^{NV_N(z)} = \frac{\sinh Nz\frac{\pi}{2\gamma}}{\sinh Nz\left(\frac{\pi}{2\gamma} - 1\right)} e^{-N\zeta z}. \tag{7.24}$$

In particular,

$$e^{NV_N(0)} = \frac{\pi}{\pi - 2\gamma}. \tag{7.25}$$

Therefore,

$$\frac{e^{-NG_N(z)}}{e^{-NG_N(0)}} = \left(\frac{\pi - 2\gamma}{\pi}\right) e^{-2N[g_N(z) - g_N(0)] + NV_N(z)}, \tag{7.26}$$

so that

$$e^{-NG_N(z)} = C_N e^{-2N[g_N(z) - g_N(0)]} \left[\frac{\sinh Nz\frac{\pi}{2\gamma}}{\sinh Nz\left(\frac{\pi}{2\gamma} - 1\right)} e^{-N\zeta z}\right], \tag{7.27}$$

where

$$C_N = \frac{\pi - 2\gamma}{\pi} e^{-NG_N(0)} \tag{7.28}$$

By (7.20),

$$C_N = \frac{\pi - 2\gamma}{\pi} e^{-N\pi i(1+\zeta)}(1 + O(N^{-1})). \tag{7.29}$$

Observe that if $\operatorname{Im} z > 0$ then

$$g_N(z) - g_N(0) = \int_0^z \omega_N(s)ds, \tag{7.30}$$

where the integration is taken over the interval $[0, z]$. From (5.56) we obtain that

$$g_N(z) - g_N(0) = -\frac{iz \ln N}{\pi} + \int_0^z b(s)ds - \frac{1}{2\pi^2}\int_0^z m_0(Ns)ds + O(N^{-2}|z|). \tag{7.31}$$

In particular, for $z = iN^{-1}y$, where $y > 0$ is bounded, we obtain that

$$g_N(iN^{-1}y) - g_N(0) = \frac{yN^{-1}\ln N}{\pi} + ib(+i0)yN^{-1} - \frac{1}{2\pi^2}M_0(iy)N^{-1} + O(N^{-2}), \tag{7.32}$$

where

$$M_0(z) = \int_0^z m_0(s)ds. \tag{7.33}$$

Thus, (7.27) gives that

$$e^{-NG_N(iN^{-1}y)} = e^{iN\omega}k_N(y)\frac{\sin y\frac{\pi}{2\gamma}}{\sin y\left(\frac{\pi}{2\gamma} - 1\right)}(1 + O(N^{-1})), \qquad y > 0, \tag{7.34}$$

where

$$\omega = -\pi(1 + \zeta), \tag{7.35}$$

and

$$k_N(y) = \frac{\pi - 2\gamma}{\pi}e^{\varphi(y)}N^{-\frac{2y}{\pi}}, \tag{7.36}$$

with

$$\varphi(y) = -i2b(+i0)y + \frac{1}{\pi^2}M_0(iy) - i\zeta y. \tag{7.37}$$

By using the value of $b(+i0)$ given in (5.60), we obtain that

$$\varphi(y) = -iy - \frac{2y}{\pi}\ln\left(2\pi\cos\frac{\pi\zeta}{2}\right) + \frac{1}{\pi^2}M_0(iy). \tag{7.38}$$

From (7.33) and (5.58) we obtain that

$$\frac{1}{\pi^2}M_0(iy) = Q(iy) - Q(+i0), \tag{7.39}$$

where

$$Q(z) := \frac{i}{\pi}\left[\int_{-\infty}^{\infty}\log(z - \mu)f(\mu)d\mu - 2z\log z + 2z\right]. \tag{7.40}$$

Since $f(\mu)$ is odd, we have that

$$Q(iy) - Q(+i0) = \frac{2}{\pi}\left[\int_0^{\infty}\arg(iy + \mu)f(\mu)d\mu + y\ln y - y\right] + iy. \tag{7.41}$$

Thus,

$$\varphi(y) = -\frac{2y}{\pi}\ln\left(2\pi\cos\frac{\pi\zeta}{2}\right) + \frac{2}{\pi}\left[\int_0^{\infty}\arg(iy + \mu)f(\mu)d\mu + y\ln y - y\right]. \tag{7.42}$$

Consider now $\operatorname{Im} z < 0$. Similar to (7.22) we have that $G_N(\mu)$ is analytically continued to $G_N(z)$ with $\operatorname{Im} z < 0$ as

$$G_N(z) = -2g_N(z) + V_N(z) + l_N, \qquad \operatorname{Im} z < 0. \tag{7.43}$$

From (7.2),

$$g_N(\bar{z}) = \overline{g_N(z)}. \tag{7.44}$$

Also, $V_N(\bar z)=\overline{V_N(z)}$ and $l_N\in\mathbb{R}$, hence

$$G_N(\bar z)=-\overline{G_N(z)}. \tag{7.45}$$

From (7.34) we obtain now that

$$e^{NG_N(-iN^{-1}y)}=e^{-N\overline{G_N(iN^{-1}y)}}=e^{-iN\omega}k_N(y)\frac{\sin y\frac{\pi}{2\gamma}}{\sin y\left(\frac{\pi}{2\gamma}-1\right)}(1+O(N^{-1})),\qquad y>0. \tag{7.46}$$

Model RH problem. Note that the jump matrix $J_S(z)$ converges, as $N\to\infty$, to the identity matrix, except on the interval $[\alpha,\beta]$ where it is constant. This leads to the following model RH problem.

(i) $M(z)$ is analytic on $\mathbb{C}\setminus[\alpha,\beta]$.

(ii) *(jump condition)*

$$M_+(z)=M_-(z)J_M,\quad z\in[\alpha_N,\beta_N], \tag{7.47}$$

where

$$J_M=\begin{pmatrix}0&1\\-1&0\end{pmatrix}. \tag{7.48}$$

(iii) *(asymptotics at infinity)*

$$M(z)=I+O\left(|z|^{-1}\right),\quad |z|\to\infty. \tag{7.49}$$

Solution to the model RH problem. The model RH problem can be solved explicitly. Namely, let us reduce it to a pair of scalar RH problems that are solved by the Plemelj-Sohotski formula. By diagonalizing the matrix J_M, we have that

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}=\frac12\begin{pmatrix}1&1\\i&-i\end{pmatrix}\begin{pmatrix}i&0\\0&-i\end{pmatrix}\begin{pmatrix}1&-i\\1&i\end{pmatrix} \tag{7.50}$$

Let

$$\tilde M(z)=\begin{pmatrix}1&-i\\1&i\end{pmatrix}M(z)\begin{pmatrix}1&1\\i&-i\end{pmatrix}. \tag{7.51}$$

Then, clearly

(i) $\tilde M(z)$ is analytic on $\mathbb{C}\setminus[\alpha_N,\beta_N]$.

(ii)

$$\tilde M_+(z)=\tilde M_-(z)\begin{pmatrix}i&0\\0&-i\end{pmatrix},\quad z\in[\alpha_N,\beta_N]. \tag{7.52}$$

(iii)

$$\tilde M(z)=I+O\left(|z|^{-1}\right),\quad |z|\to\infty. \tag{7.53}$$

Thus,

$$\begin{aligned}\tilde M(z)&=\begin{pmatrix}e^{\frac{1}{2\pi i}\int_{\alpha_N}^{\beta_N}\frac{\log i}{s-z}ds}&0\\0&e^{\frac{1}{2\pi i}\int_{\alpha_N}^{\beta_N}\frac{\log(-i)}{s-z}ds}\end{pmatrix}\\&=\begin{pmatrix}e^{\frac14\log\frac{\beta_N-z}{\alpha_N-z}}&0\\0&e^{-\frac14\log\frac{\beta_N-z}{\alpha_N-z}}\end{pmatrix}\\&=\begin{pmatrix}\gamma_N^{-1}&0\\0&\gamma_N\end{pmatrix},\end{aligned} \tag{7.54}$$

where

$$\gamma_N(z) = \left(\frac{z-\alpha_N}{z-\beta_N}\right)^{1/4} \tag{7.55}$$

with cut on $[\alpha_N, \beta_N]$ and the branch such that $\gamma_N(\infty) = 1$. Then

$$\begin{aligned} M(z) &= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \gamma_N^{\sigma_3} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{\gamma_N(z)+\gamma_N^{-1}(z)}{2} & \frac{\gamma_N(z)-\gamma_N^{-1}(z)}{(-2i)} \\ \frac{\gamma_N(z)-\gamma_N^{-1}(z)}{2i} & \frac{\gamma_N(z)+\gamma_N^{-1}(z)}{2} \end{pmatrix}, \quad \det M(z) = 1. \end{aligned} \tag{7.56}$$

At infinity we have that

$$M(z) = I + \frac{1}{z}\begin{pmatrix} 0 & \frac{\beta_N-\alpha_N}{-4i} \\ \frac{\beta_N-\alpha_N}{4i} & 0 \end{pmatrix} + \frac{1}{z^2}\begin{pmatrix} \frac{(\beta_N-\alpha_N)^2}{32} & \frac{\beta_N^2-\alpha_N^2}{-8i} \\ \frac{\beta_N^2-\alpha_N^2}{8i} & \frac{(\beta_N-\alpha_N)^2}{32} \end{pmatrix} + O(|z|^{-3}). \tag{7.57}$$

At the origin,

$$\gamma_N(+i0) = \left(\frac{-\alpha_N}{+i0-\beta_N}\right)^{1/4} = \sqrt{\tan\frac{\pi}{4}(1-\zeta)}\, e^{-\pi i/4} + O(N^{-2}), \tag{7.58}$$

hence

$$M(+i0) = \begin{pmatrix} p+iq & p-iq \\ -p+iq & p+iq \end{pmatrix} + O(N^{-2}), \tag{7.59}$$

where

$$p, q = \frac{\sqrt{2}}{4}\left[\sqrt{\tan\frac{\pi}{4}(1+\zeta)} \pm \sqrt{\tan\frac{\pi}{4}(1-\zeta)}\right]. \tag{7.60}$$

We have the conjugation condition,

$$M(\bar{z}) = \sigma_3 \overline{M(z)} \sigma_3, \tag{7.61}$$

hence

$$M(-i0) = \begin{pmatrix} p-iq & -p-iq \\ p+iq & p-iq \end{pmatrix} + O(N^{-2}). \tag{7.62}$$

8. Parametrix at the edge points

We consider small disks $D(\beta_N, r)$, $D(\alpha_N, r)$ of radius $r > 0$, centered at the edge points, and we look for a local parametrix P defined on $D(\beta_N, r) \cup D(\alpha_N, r)$ such that

(i) $P(z)$ is analytic on $(D(\beta_N, r) \cup D(\alpha_N, r)) \setminus (\mathbb{R} \cup \Sigma_N)$, where $\Sigma_N = \Sigma_N^+ \cup \Sigma_N^-$ is the boundary of the lenses, see Figure 6.

(ii) $P_+(z) = P_-(z) J_S(z), \quad z \in (D(\beta_N, r) \cup D(\alpha_N, r)) \cap (\mathbb{R} \cup \Sigma)$.

(iii) $P(z) = (I + O(N^{-1}))\, M(z), \quad z \in \partial D(\beta_N, r) \cup \partial D(\alpha_N, r), \quad N \to \infty$.

We consider the right edge point β_N in detail. Note that by (3.17), we have that for $z \in D(\beta_N, r)$,

$$\begin{aligned} -g_N(z) + \frac{V_N(z)}{2} + \frac{l_N}{2} &= \frac{1}{2}\int_{\beta_N}^{z} h_N(\mu)\sqrt{(\mu-\alpha_N)(\mu-\beta_N)}\, d\mu \\ &= \frac{2}{3} a_N(z)(z-\beta_N)^{3/2}, \quad z \in D(\beta_N, r) \setminus [\alpha_N, \beta_N], \end{aligned} \tag{8.1}$$

where $a_N(z)$ is an analytic function in $D(\beta_N, r)$ such that

$$a_N(\beta_N) = \frac{1}{2}\, h_N(\beta_N)\sqrt{\beta_N - \alpha_N} = \frac{2}{\beta\sqrt{\beta-\alpha}} + O(N^{-2}) \, > 0. \tag{8.2}$$

Define the analytic function,

$$\lambda_N(z) = \left[\frac{3}{2}\left(-g_N(z) + \frac{V_N(z)}{2} + \frac{l_N}{2}\right)\right]^{2/3} = a_N(z)^{2/3}(z-\beta_N), \tag{8.3}$$

so that $\lambda_N'(\beta_N) = a_N(\beta_N)^{2/3} > 0$, and consider the conformal mapping,

$$\lambda_N : \; D(\beta_N, r) \to \mathbb{C}. \tag{8.4}$$

We will assume that the contours $\Sigma_N^{\pm}$ are chosen in $D(\beta_N, r)$ in such a way that

$$\lambda_N : \; \Sigma_N^{\pm} \to \left\{ z : \; \arg z = \pm\frac{2\pi}{3} \right\}. \tag{8.5}$$

Let us transform the RH problem on the matrix $S(z)$ in $D(\beta_N, r)$. Let

$$\Phi(z) = S(z) e^{N\left(g_N(z) - \frac{V_N(z)}{2} - \frac{l_N}{2}\right)\sigma_3}. \tag{8.6}$$

Lemma 8.1. $\Phi(z)$ *satisfies the jump condition*

$$\Phi_+(z) = \Phi_-(z) J_\Phi, \tag{8.7}$$

where

$$J_\Phi = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & \text{for } \arg z = 0, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } z \in \Sigma_N^+, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{for } \arg z = \pi, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } z \in \Sigma_N^-. \end{cases} \tag{8.8}$$

We will use a model solution to (8.7), which is constructed explicitly in a standard way out of the Airy functions. The Airy function $\mathrm{Ai}(z)$ solves the equation $y'' = zy$ and for any $\varepsilon > 0$, in the sector $\pi + \varepsilon \le \arg z \le \pi - \varepsilon$, it has the asymptotics as $z \to \infty$,

$$\begin{aligned} \mathrm{Ai}(z) &= \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\frac{2}{3}z^{3/2}} \left(1 - \frac{5}{48} z^{-3/2} + \frac{385}{4608} z^{-3} + O(z^{-9/2})\right), \\ \mathrm{Ai}'(z) &= -\frac{1}{2\sqrt{\pi}} z^{1/4} e^{-\frac{2}{3}z^{3/2}} \left(1 + \frac{7}{48} z^{-3/2} - \frac{455}{4608} z^{-3} + O(z^{-9/2})\right). \end{aligned} \tag{8.9}$$

The functions $\mathrm{Ai}(\omega z)$, $\mathrm{Ai}(\omega^2 z)$, where $\omega = e^{\frac{2\pi i}{3}}$, also solve the equation $y'' = zy$, and we have the linear relation,

$$\mathrm{Ai}(z) + \omega \mathrm{Ai}(\omega z) + \omega^2 \mathrm{Ai}(\omega^2 z) = 0. \tag{8.10}$$

Write

$$y_0(z) = \mathrm{Ai}(z), \quad y_1(z) = \omega \mathrm{Ai}(\omega z), \quad y_2(z) = \omega^2 \mathrm{Ai}(\omega^2 z), \tag{8.11}$$

and we use these functions to define

$$\Phi(z)=\begin{cases}\begin{pmatrix} y_0(z) & -y_2(z) \\ y_0'(z) & -y_2'(z)\end{pmatrix}, & \text{for } 0<\arg z<2\pi/3, \\ \begin{pmatrix} -y_1(z) & -y_2(z) \\ -y_1'(z) & -y_2'(z)\end{pmatrix}, & \text{for } 2\pi/3<\arg z<\pi, \\ \begin{pmatrix} -y_2(z) & y_1(z) \\ -y_2'(z) & y_1'(z)\end{pmatrix}, & \text{for } -\pi<\arg z<-2\pi/3, \\ \begin{pmatrix} y_0(z) & y_1(z) \\ y_0'(z) & y_1'(z)\end{pmatrix}, & \text{for } -2\pi/3<\arg z<0. \end{cases} \tag{8.12}$$

Then in the sector $0<\arg z<2\pi/3$,

$$\Phi(z)=\begin{pmatrix} \Phi_{11}(z) & \Phi_{12}(z) \\ \Phi_{21}(z) & \Phi_{22}(z)\end{pmatrix}, \tag{8.13}$$

where

$$\begin{aligned}\Phi_{11}(z)&=\frac{1}{2\sqrt{\pi}}z^{-1/4}e^{-\frac{2}{3}z^{3/2}}\left(1-\frac{5}{48}z^{-3/2}+O(z^{-3})\right),\\ \Phi_{12}(z)&=\frac{1}{2\sqrt{\pi}}(-\omega^2)(\omega^2 z)^{-1/4}e^{-\frac{2}{3}(\omega^2 z)^{3/2}}\left(1-\frac{5}{48}(\omega^2 z)^{-3/2}+O(z^{-3})\right),\\ \Phi_{21}(z)&=-\frac{1}{2\sqrt{\pi}}z^{-1/4}e^{-\frac{2}{3}z^{3/2}}\left(1+\frac{7}{48}z^{-3/2}+O(z^{-3})\right),\\ \Phi_{22}(z)&=\frac{1}{2\sqrt{\pi}}\omega(\omega^2 z)^{1/4}e^{-\frac{2}{3}(\omega^2 z)^{3/2}}\left(1+\frac{7}{48}(\omega^2 z)^{-3/2}+O(z^{-3})\right).\end{aligned} \tag{8.14}$$

where for $z^{-1/4}$, $z^{1/4}$, and $z^{3/2}$ the principal branches are taken, with the cut on $(-\infty,0)$. Since $\omega^2=e^{\frac{4\pi i}{3}}$ and $0<\arg z<\frac{2\pi}{3}$, we have that $\arg\omega^2 z=\arg z-\frac{2\pi}{3}$, hence $(\omega^2 z)^{1/4}=e^{-\frac{\pi i}{6}}z^{1/4}$, $(\omega^2 z)^{-1/4}=e^{\frac{\pi i}{6}}z^{-1/4}$, $(\omega^2 z)^{3/2}=-z^{3/2}$, and $(\omega^2 z)^{-3/2}=-z^{-3/2}$. Substituting these expressions into (8.13), we obtain that

$$\Phi(z)=\frac{1}{2\sqrt{\pi}}z^{-\sigma_3/4}\left[\begin{pmatrix} 1 & i \\ -1 & i\end{pmatrix}+\frac{1}{48}\begin{pmatrix} -5 & 5i \\ -7 & -7i\end{pmatrix}z^{-3/2}+O(z^{-3})\right]e^{-\frac{2}{3}z^{3/2}\sigma_3} \tag{8.15}$$

Note that $\Phi(z)$ satisfies the jump condition $\Phi_+(z)=\Phi_-(z)J_\Phi$. Define

$$P(z)=E(z)N^{\frac{1}{6}\sigma_3}\Phi(N^{2/3}\lambda_N(z))e^{N\left(-g_N(z)+\frac{V_N(z)}{2}+\frac{l_N}{2}\right)\sigma_3}, \tag{8.16}$$

where $E(z)$ is an analytic prefactor that has to be chosen to satisfy the matching condition $P(z)=\left(I+O\left(N^{-1}\right)\right)M(z)$ on the boundary of $D(\beta_N,r)$. Then

$$\begin{aligned}E(z)&=\sqrt{\pi}M(z)\begin{pmatrix} 1 & -1 \\ -i & -i\end{pmatrix}(\lambda_N(z))^{\sigma_3/4}\\ &=\sqrt{\pi}\begin{pmatrix} 1 & -1 \\ -i & -i\end{pmatrix}\begin{pmatrix} \gamma_N(z)\lambda_N^{1/4}(z) & 0 \\ 0 & \gamma_N^{-1}(z)\lambda_N^{-1/4}(z)\end{pmatrix}.\end{aligned} \tag{8.17}$$

Recall the definition of $\gamma_N(z) = \left(\frac{z-\alpha_N}{z-\beta_N}\right)^{1/4}$ and note that

$$\gamma_N(z)\lambda_N^{1/4}(z) = (z-\alpha_N)^{1/4}(a_N(z))^{1/6}.$$

Therefore $E(z)$ is indeed an analytic function in $D(\beta_N, r)$.

A similar construction works for a parametrix P around the other edge point. Namely, by (3.18), we have that for $z \in D(\alpha_N, r)$,

$$\begin{aligned} -g_N(z) + \frac{V_N(z)}{2} + \frac{l_N}{2} + \pi i \,\mathrm{sgn}\,(\mathrm{Im}\, z) &= \frac{1}{2}\int_z^{\alpha_N} h_N(\mu)\sqrt{(\alpha_N-\mu)(\beta_N-\mu)}\, d\mu \\ &= \frac{2}{3}a_N(z)(\alpha_N - z)^{3/2}, \quad z \in D(\alpha_N, r)\setminus[\alpha_N, \beta_N], \end{aligned} \tag{8.18}$$

where $a_N(z)$ is an analytic function in $D(\alpha_N, r)$ such that

$$a_N(\alpha_N) = \frac{1}{2}\, h(\alpha_N)\sqrt{\beta_N - \alpha_N} = \frac{2}{(-\alpha)\sqrt{\beta-\alpha}} + O(N^{-2}) \, > 0. \tag{8.19}$$

Define the analytic function,

$$\lambda_N(z) = \left[\frac{3}{2}\left(-g_N(z) + \frac{V_N(z)}{2} + \frac{l_N}{2} + \pi i \,\mathrm{sgn}\,(\mathrm{Im}\, z)\right)\right]^{2/3} = a_N(z)^{2/3}(\alpha_N - z), \tag{8.20}$$

so that $\lambda_N'(\alpha_N) = -a_N(\alpha_N)^{2/3} < 0$, and then define $P(z)$ by the formula,

$$P(z) = \sigma_3 E(z) N^{\frac{1}{6}\sigma_3}\Phi(N^{2/3}\lambda_N(z)) e^{N\left(-g_N(z)+\frac{V_N(z)}{2}+\frac{l_N}{2}\right)\sigma_3}\sigma_3, \tag{8.21}$$

where

$$\begin{aligned} E(z) &= \sqrt{\pi}\sigma_3 M(z)\sigma_3 \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} (\lambda_N(z))^{\sigma_3/4} \\ &= \sqrt{\pi}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}\begin{pmatrix} \gamma_N^{-1}(z)\lambda_N^{1/4}(z) & 0 \\ 0 & \gamma_N(z)\lambda_N^{-1/4}(z) \end{pmatrix}. \end{aligned} \tag{8.22}$$

Observe that the function

$$\gamma_N^{-1}(z)\lambda_N^{1/4}(z) = (\beta_N - z)^{1/4}(a_N(z))^{1/6}.$$

is analytic in $D(\alpha_N, r)$, hence $E(z)$ is analytic as well.

9. Approximate solution to the RH problem

Define

$$R(z) = \begin{cases} S(z)P^{-1}(z), & \text{if } z \in D(\alpha_N, r) \cup D(\beta_N, r), \\ S(z)M^{-1}(z), & \text{otherwise.} \end{cases} \tag{9.1}$$

Then, in $D(\alpha_N, r) \cup D(\beta_N, r)$ we have that

$$\begin{aligned} R_+(z) &= S_+(z)P_+^{-1}(z) = S_-(z)J_S(z)J_S^{-1}(z)P_-^{-1}(z) = S_-(z)P_-^{-1}(z) \\ &= R_-(z), \end{aligned} \tag{9.2}$$

on $\Sigma_N^+ \cup \Sigma_N^- \cup (\mathbb{R}\setminus[\alpha_N - r, \beta_N + r])$,

$$\begin{aligned} R_+(z) &= S_+(z)M^{-1}(z) = S_-(z)J_S(z)M^{-1}(z) = S_-(z)M^{-1}(z)M(z)J_S(z)M^{-1}(z) \\ &= R_-(z)M(z)J_S(z)M^{-1}(z), \end{aligned} \tag{9.3}$$

on $[\alpha_N + r, \beta_N - r]$,

$$\begin{aligned} R_+(z) = S_+(z)M_+^{-1}(z) = S_-(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} M_-^{-1}(z) = S_-(z)M_-^{-1}(z) \\ = R_-(z), \end{aligned} \tag{9.4}$$

and on $\partial D(\alpha_N, r) \cup \partial D(\beta_N, r)$ the jump matrix is

$$J_R(z) = R_-^{-1}(z)R_+(z) = P(z)S^{-1}(z)S(z)M^{-1}(z) = P(z)M^{-1}(z). \tag{9.5}$$

Introduce the contour Σ_R, which consists of the six arcs,

$$\Sigma_R = (-\infty, \alpha_N - r) \cup \Sigma_R^\alpha \cup \Sigma_R^+ \cup \Sigma_R^- \cup \Sigma_R^\beta \cup (\beta_N + r, \infty), \tag{9.6}$$

where

$$\Sigma_R^\alpha = \partial D(\alpha_N, r), \qquad \Sigma_R^\beta = \partial D(\beta_N, r), \qquad \Sigma_R^\pm = \Sigma_N^\pm \setminus [D(\alpha_N, r) \cup D(\beta_N, r)]. \tag{9.7}$$

see Fig. 7. The orientation of the arcs is shown on Fig. 7.

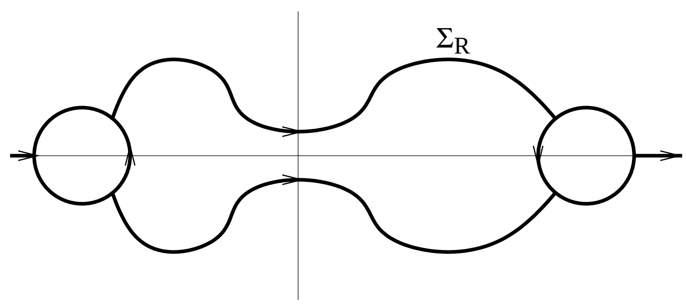

FIGURE 7. The contour Σ_R.

For the sake of brevity we will denote

$$\Sigma_R^\infty = (-\infty, \alpha_N - r) \cup (\beta_N + r, \infty). \tag{9.8}$$

We have the following

Lemma 9.1. *$S(z)$ is a solution of the Riemann-Hilbert problem (7.16) - (7.18) if and only if $R(z)$ is a solution of the following RH problem:*

(i) *$R(z)$ is analytic on $\mathbb{C} \setminus \Sigma_R$,*
(ii) *$R_+(z) = R_-(z)J_R(z)$, $z \in \Sigma_R$, where*

$$J_R(z) = \begin{cases} M(z)J_S(z)M^{-1}(z), & \text{on} \quad \Sigma_R \setminus (\partial D(\alpha_N, r) \cup \partial D(\beta_N, r)), \\ P(z)M^{-1}(z), & \text{on} \quad \partial D(\alpha_N, r) \cup \partial D(\beta_N, r). \end{cases} \tag{9.9}$$

(iii) *$R(z) = I + O(z^{-1}), \quad z \to \infty.$*

We evaluate the jump matrix J_R on different pieces of Σ_R.

Jump matrix $J_R(z)$ on Σ_R^β. We have on $\partial D(\beta_N, r)$ that

$$\begin{aligned} J_R(z) &= P(z)M^{-1}(z) = E(z)N^{\frac{1}{6}\sigma_3}\Phi(N^{2/3}\lambda_N(z))e^{N(-g_N(z)+\frac{V_N(z)}{2}+\frac{l_N}{2})\sigma_3}M^{-1}(z) \\ &= \sqrt{\pi}M(z)\begin{pmatrix}1 & -1\\ -i & -i\end{pmatrix}\lambda_N^{\frac{\sigma_3}{4}}(z)N^{\frac{1}{6}}\frac{1}{2\sqrt{\pi}}N^{-\frac{1}{6}}\lambda_N^{-\frac{\sigma_3}{4}}(z)\left[\begin{pmatrix}1 & i\\ -i & i\end{pmatrix}+N^{-1}\frac{1}{48}\right. \\ &\times\left.\begin{pmatrix}-5 & 5i\\ -7 & -7i\end{pmatrix}\lambda_N^{-3/2}(z)+O\left(N^{-2}\right)\right]e^{-\frac{2}{3}N\lambda_N^{3/2}(z)\sigma_3}e^{N(-g_N(z)+\frac{V_N(z)}{2}+\frac{l_N}{2})\sigma_3}M^{-1}(z) \\ &= M(z)\left[I+\frac{1}{48}N^{-1}\begin{pmatrix}1 & 6i\\ 6i & -1\end{pmatrix}\lambda_N^{-3/2}(z)+O\left(N^{-2}\right)\right]M^{-1}(z) \\ &= I+N^{-1}J_R^1(z)+O\left(N^{-2}\right), \end{aligned} \tag{9.10}$$

where

$$J_R^1(z) = \frac{1}{96(z-\beta)^2(z-\alpha)^{1/2}a(z)}\begin{pmatrix}-5(z-\alpha)+7(z-\beta) & i[5(z-\alpha)+7(z-\beta)]\\ i[5(z-\alpha)+7(z-\beta)] & 5(z-\alpha)-7(z-\beta)\end{pmatrix}, \tag{9.11}$$

and

$$a(z) = \frac{3}{4(z-\beta)^{3/2}}\int_\beta^z h(s)\sqrt{(s-\alpha)(s-\beta)}\,ds, \tag{9.12}$$

where $h(s)$ is defined in (4.16).

Jump matrix $J_R(z)$ on Σ_R^α. Similarly, on $\partial D(\alpha_N, r)$,

$$\begin{aligned} J_R(z) &= M(z)\left[I+\frac{1}{48}N^{-1}\begin{pmatrix}1 & -6i\\ -6i & -1\end{pmatrix}\lambda_N^{-3/2}(z)+O\left(N^{-2}\right)\right]M^{-1}(z) \\ &= I+N^{-1}J_R^1(z)+O\left(N^{-2}\right), \end{aligned} \tag{9.13}$$

where

$$J_R^1(z) = \frac{1}{96(\alpha-z)^2(\beta-z)^{1/2}a(z)}\begin{pmatrix}7(\alpha-z)-5(\beta-z) & i[-7(\alpha-z)-5(\beta-z)]\\ i[-7(\alpha-z)-5(\beta-z)] & -7(\alpha-z)+5(\beta-z)\end{pmatrix}, \tag{9.14}$$

and

$$a(z) = \frac{3}{4(\alpha-z)^{3/2}}\int_z^\alpha h(s)\sqrt{(\alpha-s)(\beta-s)}\,ds, \tag{9.15}$$

where $h(s)$ is defined in (4.17).

Jump matrix $J_R(z)$ on $\Sigma_R^\pm$. By (9.9) and (7.17), on Σ_R^+,

$$J_R(z) = I + J_R^\circ(z), \tag{9.16}$$

where

$$J_R^\circ(z) = e^{-NG_N(z)}M(z)\sigma_+M(z)^{-1}, \qquad z\in\Sigma_R^+, \tag{9.17}$$

and

$$\sigma_+ = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \qquad \sigma_- = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}. \tag{9.18}$$

From (7.26), (7.31), we obtain that there exist constants $C, \gamma, c > 0$ such that

$$\|J_R^\circ(z)\| \le CN^{-\gamma}e^{-cN|\operatorname{Im} z|}, \tag{9.19}$$

where $\|J_R^\circ(z)\|$ is the sum of absolute values of the matrix elements of $J_R^\circ(z)$. On Σ_R^- we also have equation (9.16) with estimate (9.19).

Jump matrix $J_R(z)$ on Σ_R^∞. By (7.17), $J_R(x) = I + J_R^\circ(x)$, where

$$J_R^\circ(z) = e^{N[g_{N+}(z)+g_{N-}(z)-V_N(z)-l_N]}M(z)\sigma_- M(z)^{-1}, \qquad z \in \Sigma_R^\infty. \tag{9.20}$$

In this case, there exist $C, c > 0$ such that

$$\|J_R^\circ(z)\| \le Ce^{-cN|z|}. \tag{9.21}$$

Solution of the RH problem for R by perturbation theory. The estimates above show that $J_R^\circ(z) \to 0$ as $N \to \infty$. We can apply the following general result.

Proposition 9.2. *Assume that $v(\lambda)$, $\lambda \in \Sigma_R$, solves the equation*

$$v(\lambda) = I - \frac{1}{2\pi i}\int_{\Sigma_R} \frac{v(\mu)J_R^\circ(\mu)}{\lambda_- - \mu}\,d\mu, \quad \lambda \in \Sigma_R, \tag{9.22}$$

where λ_- means $\lambda - i0$, the value of the limit from the minus side, and $J_R = I + J_R^\circ$. Then

$$R(z) = I - \frac{1}{2\pi i}\int_{\Sigma_R} \frac{v(\mu)J_R^\circ(\mu)}{z - \mu}\,d\mu, \quad z \in \mathbb{C}\setminus\Sigma_R, \tag{9.23}$$

solves the following RH problem:

(i) *$R(z)$ is analytic on $\mathbb{C}\setminus\Sigma_R$,*
(ii) *$R_+(\lambda) = R_-(\lambda)J_R(\lambda)$, $\lambda \in \Sigma_R$,*
(iii) *$R(z) = I + O\left(z^{-1}\right), \quad z \to \infty$.*

Proof. From (9.22), (9.23),

$$R_-(\lambda) = v(\lambda), \quad \lambda \in \Sigma_R. \tag{9.24}$$

By the jump property of the Cauchy transform,

$$R_+(\lambda) - R_-(\lambda) = v(\lambda)J_R^\circ(\lambda) = R_-(\lambda)J_R^\circ(\lambda), \tag{9.25}$$

hence $R_+(\lambda) = R_-(\lambda)J_R(\lambda)$. From (9.23), $R(z) = I + O(z^{-1})$. Proposition 9.2 is proved.

Equation (9.22) can be solved by perturbation theory, so that

$$v(\lambda) = I + \sum_{k=1}^\infty v_k(\lambda), \tag{9.26}$$

where for $k \ge 1$,

$$v_k(\lambda) = -\frac{1}{2\pi i}\int_{\Sigma_R} \frac{v_{k-1}(\mu)J_R^\circ(\mu)}{\lambda_- - \mu}\,d\mu, \quad \lambda \in \Sigma_R, \tag{9.27}$$

and $v_0(\lambda) = I$. Series (9.26) is estimated from above by a convergent geometric series, so it is absolutely convergent. Observe that

$$v_1(\lambda) = -\frac{1}{2\pi i}\int_{\Sigma_R} \frac{J_R^\circ(\mu)}{\lambda_- - \mu}\,d\mu, \quad \lambda \in \Sigma_R, \tag{9.28}$$

The function $R(z)$ is given then as

$$R(z) = I + \sum_{k=1}^\infty R_k(z), \tag{9.29}$$

where

$$R_k(z) = -\frac{1}{2\pi i}\int_{\Sigma_R} \frac{v_{k-1}(\mu)J_R^\circ(\mu)}{z-\mu}\,d\mu. \tag{9.30}$$

In particular,

$$R_1(z) = -\frac{1}{2\pi i}\int_{\Sigma_R} \frac{J_R^\circ(\mu)}{z-\mu}\,d\mu. \tag{9.31}$$

10. Large N asymptotics of the recurrent coefficients

From (6.9), (7.1) and (7.15), we obtain the formulae for the recurrent coefficients:

$$\begin{aligned} h_{NN} &= -2\pi i e^{Nl_N}(S_1)_{12},\\ R_{NN} &= (S_1)_{21}(S_1)_{12},\\ Q_{NN} &= \frac{(S_2)_{21}}{(S_2)_{21}} + (S_1)_{11}. \end{aligned} \tag{10.1}$$

where

$$S(z) = I + \frac{S_1}{z} + \frac{S_2}{z^2} + O(z^{-3})\,, \qquad z\to\infty. \tag{10.2}$$

By (9.1), $S(z) = R(z)M(z)$ for large z, hence

$$S_1 = M_1 + R_1, \tag{10.3}$$

where

$$M(z) = I + \frac{M_1}{z} + O(z^{-2}), \qquad R(z) = I + \frac{R_1}{z} + O(z^{-2}). \tag{10.4}$$

Therefore,

$$R_{NN} = (M_1+R_1)_{21}(M_1+R_1)_{12}. \tag{10.5}$$

By (7.57),

$$M_1 = \begin{pmatrix} 0 & \frac{\beta_N-\alpha_N}{-4i} \\ \frac{\beta_N-\alpha_N}{4i} & 0 \end{pmatrix}, \tag{10.6}$$

hence

$$R_{NN} = \left(\frac{\beta_N-\alpha_N}{4}\right)^2 + \frac{\beta_N-\alpha_N}{4i}[(R_1)_{12}-(R_1)_{21}] + (R_1)_{12}(R_1)_{21}. \tag{10.7}$$

By (9.23),

$$\begin{aligned} R_1 &= -\frac{1}{2\pi i}\int_{\Sigma_R} v(\lambda)J_R^\circ(\lambda)d\lambda\\ &= -\frac{1}{2\pi i}\int_{\Sigma_R} J_R^\circ(\lambda)d\lambda - \frac{1}{2\pi i}\int_{\Sigma_R} v_1(\lambda)J_R^\circ(\lambda)d\lambda - \dots \end{aligned} \tag{10.8}$$

We will call the first term on the right,

$$R_1^{(1)} = -\frac{1}{2\pi i}\int_{\Sigma_R} J_R^\circ(\lambda)d\lambda, \tag{10.9}$$

a linear term, the second one,

$$R_1^{(2)} = -\frac{1}{2\pi i}\int_{\Sigma_R} v_1(\lambda)J_R^\circ(\lambda)d\lambda, \tag{10.10}$$

a quadratic term, etc. By definition, we have that

$$R_1 = R_1^{(1)} + R_1^{(2)} + \dots \tag{10.11}$$

First we evaluate the linear term.

Evaluation of the linear term. Denote

$$R_1^a = -\frac{1}{2\pi i}\int_{\Sigma_R^a} J_R^\circ(\lambda)d\lambda, \qquad a=\alpha,\beta,+,-,\infty, \tag{10.12}$$

so that

$$R_1^{(1)} = R_1^\alpha + R_1^\beta + R_1^+ + R_1^- + R_1^\infty. \tag{10.13}$$

Let us evaluate R_1^α, R_1^β, $R_1^\pm$, and R_1^∞.

Evaluation of R_1^α, R_1^β. By (9.10),

$$R_1^\beta = -\frac{N^{-1}}{2\pi i}\oint_{\partial D(\beta_N,r)} J_R^1(z)\,dz + O(N^{-2}), \tag{10.14}$$

which can be evaluated by taking the residue at $z=\beta$. The result is

$$R_1^\beta = -N^{-1}\frac{1}{192}\begin{pmatrix} 3\beta+\alpha & i(11\beta-\alpha) \\ i(11\beta-\alpha) & -3\beta-\alpha\end{pmatrix} + O(N^{-2}). \tag{10.15}$$

A similar expression holds for R_1^α. Namely,

$$R_1^\alpha = -N^{-1}\frac{1}{192}\begin{pmatrix} -3\alpha-\beta & i(11\alpha-\beta) \\ i(11\alpha-\beta) & 3\alpha+\beta\end{pmatrix} + O(N^{-2}). \tag{10.16}$$

By taking into account terms of the order of N^{-2} in (9.10), we obtain the error terms in (10.15), (10.16) as $N^{-2}c_{\alpha,\beta} + O(N^{-3})$, where $c_{\alpha,\beta}$ are some explicit matrices.

Evaluation of $R_1^\pm$. In the usual case of a random matrix model with an analytic potential $V(M)$ independent of N, the terms $R_1^\pm$, which represent the integral over the lenses boundary, are exponentially small as $N\to\infty$, see [9]. It is not the case in our situation because of the series of poles of the function $e^{-NG_N(z)}$ on the imaginary axis. By (9.17),

$$R_1^+ = -\frac{1}{2\pi i}\int_{\Sigma_R^+} J_R^\circ(\lambda)d\lambda = -\frac{1}{2\pi i}\int_{\Sigma_R^+} e^{-NG_N(\lambda)}M(\lambda)\sigma_+M(\lambda)^{-1}d\lambda. \tag{10.17}$$

From (7.27) we obtain that the function $e^{-NG_N(z)}$ has simple poles at the points

$$z = z_j = iN^{-1}y_j, \qquad y_j = \frac{j\pi}{\frac{\pi}{2\gamma}-1}\,, \qquad j=1,2,\dots, \tag{10.18}$$

and by (7.34), the residue at z_j is equal to

$$\operatorname*{Res}_{z=z_j}\left[e^{-NG_N(z)}\right] = e^{iN\omega}k_N(y_j)\frac{i(-1)^j\sin y_j\frac{\pi}{2\gamma}}{N\left(\frac{\pi}{2\gamma}-1\right)}(1+O(N^{-1})), \tag{10.19}$$

By using (7.36) we reduce this to

$$\operatorname*{Res}_{z=z_j}\left[e^{-NG_N(z)}\right] = iC_je^{iN\omega}N^{-\kappa_j}(1+O(N^{-1})), \tag{10.20}$$

where

$$\kappa_j = 1 + \frac{2j}{\frac{\pi}{2\gamma} - 1}\,, \qquad C_j = \frac{2\gamma}{\pi} e^{\varphi(y_j)} (-1)^j \sin\left(\frac{\pi j}{1 - \frac{2\gamma}{\pi}}\right). \tag{10.21}$$

Observe that $\kappa_j > 1$. From (7.46) we obtain that

$$\operatorname*{Res}_{z=-z_j}\left[e^{NG_N(z)}\right] = -iC_j e^{-iN\omega} N^{-\kappa_j}(1 + O(N^{-1})), \tag{10.22}$$

Let us deform the contour Σ_R^+ up, crossing the poles. Every time we cross a pole, the residue at the pole appears on the right of (10.17), while the integral becomes smaller than the contribution from the pole. This gives the asymptotic expansion as $N \to \infty$,

$$R_1^+ \sim -\sum_{j=1}^{\infty} \operatorname*{Res}_{z=z_j}\left[e^{-NG_N(z)}\right] M(z_j)\sigma_+ M(z_j)^{-1}, \tag{10.23}$$

where the j-th term is of the order of $N^{-\kappa_j}$. For our purposes it will be sufficient to consider terms with $\kappa_j \le 2$ only,

$$R_1^+ = -\sum_{j:\,\kappa_j \le 2} \operatorname*{Res}_{z=z_j}\left[e^{-NG_N(z)}\right] M(z_j)\sigma_+ M(z_j)^{-1} + O(N^{-2-\varepsilon}), \tag{10.24}$$

where

$$j_0 = \left[\frac{1}{2}\left(\frac{\pi}{2\gamma} - 1\right)\right]. \tag{10.25}$$

In fact, since $z_j = O(N^{-1})$, we can replace $M(z_j)$ by $M(+i0)$,

$$R_1^+ = -\sum_{j:\,\kappa_j \le 2} \operatorname*{Res}_{z=z_j}\left[e^{-NG_N(z)}\right] M(+i0)\sigma_+ M(+i0)^{-1} + O(N^{-2-\varepsilon}). \tag{10.26}$$

Let us rewrite this in terms of the matrix elements,

$$\begin{aligned} &(R_1^+)_{12} = {M_{12}^+}^2 J_{21}^+ + O(N^{-2-\varepsilon}), \qquad (R_1^+)_{21} = -{M_{22}^+}^2 J_{21}^+ + O(N^{-2-\varepsilon}),\\ &(R_1^+)_{11} = -M_{12}^+ M_{22} J_{21}^+ + O(N^{-2-\varepsilon}), \qquad (R_1^+)_{22} = M_{12}^+ M_{22} J_{21}^+ + O(N^{-2-\varepsilon}), \end{aligned} \tag{10.27}$$

where

$$J_{21}^+ = \sum_{j:\,\kappa_j \le 2} \operatorname*{Res}_{z=z_j}\left[e^{-NG_N(z)}\right] = ie^{iN\omega} \sum_{j:\,\kappa_j \le 2} C_j N^{-\kappa_j} + O(N^{-2-\varepsilon}), \tag{10.28}$$

and M_{ij}^+ are the matrix elements of the matrix $M(+i0)$. By applying (7.59), we obtain that

$$(R_1^+)_{12} = (p - iq)^2 J_{21}^+ + O(N^{-2-\varepsilon}), \qquad (R_1^+)_{21} = -(p + iq)^2 J_{21}^+ + O(N^{-2-\varepsilon}), \tag{10.29}$$

Similarly, we evaluate the contributions from the contour Σ_R^- as

$$(R_1^-)_{12} = (p + iq)^2 J_{21}^- + O(N^{-2-\varepsilon}), \qquad (R_1^-)_{21} = -(p - iq)^2 J_{21}^- + O(N^{-2-\varepsilon}), \tag{10.30}$$

where

$$J_{21}^- = -\sum_{j:\,\kappa_j \le 2} \operatorname*{Res}_{z=-z_j}\left[e^{NG_N(z)}\right] = ie^{-iN\omega} \sum_{j:\,\kappa_j \le 2} C_j N^{-\kappa_j} + O(N^{-2-\varepsilon}). \tag{10.31}$$

By combining (10.29) and (10.30), we obtain that

$$
\begin{aligned}
(R_1^+)_{12}+(R_1^-)_{12} &= i2\left[(p^2-q^2)\cos(N\omega)+2pq\sin(N\omega)\right]\sum_{j:\,\kappa_j\le 2}C_jN^{-\kappa_j}+O(N^{-2-\varepsilon}),\\
(R_1^+)_{21}+(R_1^-)_{21} &= -i2\left[(p^2-q^2)\cos(N\omega)-2pq\sin(N\omega)\right]\sum_{j:\,\kappa_j\le 2}C_jN^{-\kappa_j}+O(N^{-2-\varepsilon})\,.
\end{aligned}
\tag{10.32}
$$

From (7.60) we find that

$$
p^2-q^2=\frac{1}{2}\,,\qquad 2pq=\frac{1}{2}\tan\frac{\pi\zeta}{2}\,,
\tag{10.33}
$$

hence

$$
\begin{aligned}
(R_1^+)_{12}+(R_1^-)_{12} &= i\left[\cos(N\omega)+\tan\left(\frac{\pi\zeta}{2}\right)\sin(N\omega)\right]\sum_{j:\,\kappa_j\le 2}C_jN^{-\kappa_j}+O(N^{-2-\varepsilon}),\\
(R_1^+)_{21}+(R_1^-)_{21} &= -i\left[\cos(N\omega)-\tan\left(\frac{\pi\zeta}{2}\right)\sin(N\omega)\right]\sum_{j:\,\kappa_j\le 2}C_jN^{-\kappa_j}+O(N^{-2-\varepsilon})\,.
\end{aligned}
\tag{10.34}
$$

Evaluation of R_1^∞. From (9.21) we obtain that R_1^∞ is exponentially small as $N\to\infty$,

$$
R_1^\infty=O(e^{-c_0N}).
\tag{10.35}
$$

Summary for the linear term. The evaluation of the linear term can be summarized as follows:

$$
\begin{aligned}
(R_1^{(1)})_{12} &= -N^{-1}\frac{5i(\beta-\alpha)}{96}+i\left[\cos(N\omega)+\tan\left(\frac{\pi\zeta}{2}\right)\sin(N\omega)\right]\sum_{j:\,\kappa_j\le 2}C_jN^{-\kappa_j}\\
&\quad +c_{12}^{(1)}N^{-2}+O(N^{-2-\varepsilon}),\\
(R_1^{(1)})_{21} &= -N^{-1}\frac{5i(\beta-\alpha)}{96}-i\left[\cos(N\omega)-\tan\left(\frac{\pi\zeta}{2}\right)\sin(N\omega)\right]\sum_{j:\,\kappa_j\le 2}C_jN^{-\kappa_j}\\
&\quad +c_{21}^{(1)}N^{-2}+O(N^{-2-\varepsilon}),
\end{aligned}
\tag{10.36}
$$

where $c_{12}^{(1)}$, $c_{21}^{(1)}$ are some constants.

Evaluation of the quadratic term. We obtain from (9.28), (10.10), that the quadratic term is equal to

$$
R_1^{(2)}=-\frac{1}{(2\pi)^2}\int_{\Sigma_R}\int_{\Sigma_R}\frac{J_R^\circ(\mu)J_R^\circ(\lambda)}{\lambda_--\mu}\,d\mu d\lambda.
\tag{10.37}
$$

We can split it as

$$
R_1^{(2)}=\sum_{a,b\in A}R_1^{a,b},\qquad A=\{\alpha,\beta,+,-,\infty\},
\tag{10.38}
$$

where

$$
R_1^{a,b}=-\frac{1}{(2\pi)^2}\int_{\Sigma_R^a}\int_{\Sigma_R^b}\frac{J_R^\circ(\mu)J_R^\circ(\lambda)}{\lambda_--\mu}\,d\mu d\lambda.
\tag{10.39}
$$

If $a \neq b$ then we can replace λ_- by λ and in this case we obtain that

$$R_1^{a,b} = -\frac{1}{(2\pi)^2} \int_{\Sigma_R^a} \int_{\Sigma_R^b} \frac{J_R^\circ(\mu) J_R^\circ(\lambda)}{\lambda - \mu}\, d\mu d\lambda, \qquad a \neq b. \tag{10.40}$$

It is tempting to say that $R_1^{b,a} = -R_1^{a,b}$, but in general it is not true, because the matrices $J_R^\circ(\lambda)$ and $J_R^\circ(\mu)$ do not commute. By (9.20) $J_R^\circ(z)$ is analytic on Σ_R^∞ and by (9.21) it is exponentially small in $N|z|$, hence $R_1^{a,b}$ is exponentially small in N, if at least one of a, b is equal to ∞,

$$|R_1^{a,b}| \leq C_0 e^{-c_0 N}, \qquad C_0, c_0 > 0; \qquad a = \infty \quad \text{or} \quad b = \infty. \tag{10.41}$$

From (9.10) we obtain that

$$\frac{1}{2\pi i} \int_{\Sigma_R^\beta} \frac{J_R^\circ(\mu)}{\lambda_- - \mu}\, d\mu = N^{-1} \operatorname*{Res}_{\mu=\beta} \left[J_R^1(\mu)\right] \frac{1}{\lambda - \beta} + O(N^{-2}), \tag{10.42}$$

hence

$$R_1^{\beta,\beta} = N^{-2} \operatorname*{Res}_{\mu=\beta} \left[J_R^1(\mu)\right] \operatorname*{Res}_{\lambda=\beta} \left[\frac{J_R^1(\lambda)}{\lambda - \beta}\right] + O(N^{-3}). \tag{10.43}$$

Similarly,

$$R_1^{\alpha,\alpha} = N^{-2} \operatorname*{Res}_{\mu=\alpha} \left[J_R^1(\mu)\right] \operatorname*{Res}_{\lambda=\alpha} \left[\frac{J_R^1(\lambda)}{\lambda - \alpha}\right] + O(N^{-3}). \tag{10.44}$$

The cross terms are evaluated as

$$\begin{aligned} R_1^{\alpha,\beta} &= -N^{-2} \frac{1}{\beta - \alpha} \operatorname*{Res}_{\mu=\beta} \left[J_R^1(\mu)\right] \operatorname*{Res}_{\lambda=\alpha} \left[J_R^1(\lambda)\right] + O(N^{-3}), \\ R_1^{\beta,\alpha} &= N^{-2} \frac{1}{\beta - \alpha} \operatorname*{Res}_{\mu=\alpha} \left[J_R^1(\mu)\right] \operatorname*{Res}_{\lambda=\beta} \left[J_R^1(\lambda)\right] + O(N^{-3}). \end{aligned} \tag{10.45}$$

Thus,

$$R_1^{\alpha,\alpha} + R_1^{\beta,\beta} + R_1^{\alpha,\beta} + R_1^{\beta,\alpha} = c_1 N^{-2} + O(N^{-3}), \tag{10.46}$$

where

$$\begin{aligned} c_1 = &\operatorname*{Res}_{\lambda=\alpha} \left[J_R^1(\lambda)\right] \operatorname*{Res}_{\lambda=\alpha} \left[\frac{J_R^1(\lambda)}{\lambda - \alpha}\right] + \operatorname*{Res}_{\lambda=\beta} \left[J_R^1(\lambda)\right] \operatorname*{Res}_{\lambda=\beta} \left[\frac{J_R^1(\lambda)}{\lambda - \beta}\right] \\ &+ \frac{1}{\beta - \alpha} \operatorname*{Res}_{\lambda=\alpha} \left[J_R^1(\lambda)\right] \operatorname*{Res}_{\lambda=\beta} \left[J_R^1(\lambda)\right] - \frac{1}{\beta - \alpha} \operatorname*{Res}_{\lambda=\beta} \left[J_R^1(\lambda)\right] \operatorname*{Res}_{\lambda=\alpha} \left[J_R^1(\lambda)\right]. \end{aligned} \tag{10.47}$$

Let us evaluate $R_1^{+,+}$. Consider

$$v_1^+(\lambda) \equiv -\frac{1}{2\pi i} \int_{\Sigma_R^+} \frac{J_R^\circ(\mu)}{\lambda_- - \mu}\, d\mu. \tag{10.48}$$

By deforming the contour of integration up, we obtain the asymptotic expansion of $v_1^+(\lambda)$ as $N \to \infty$,

$$v_1^+(\lambda) \sim -\sum_{j=1}^{\infty} \frac{1}{\lambda - z_j} \operatorname*{Res}_{z=z_j} \left[e^{-NG_N(z)}\right] M(z_j) \sigma_+ M(z_j)^{-1}. \tag{10.49}$$

Now we substitute this asymptotic expansion into the formula,

$$R_1^{+,+} = -\frac{1}{2\pi i}\int_{\Sigma_R^+} v_1^+(\lambda) J_R^\circ(\lambda)\, d\lambda, \tag{10.50}$$

and move the contour of integration up. This gives the asymptotic series,

$$\begin{aligned} R_1^{+,+} \sim \sum_{j,k=1;\, j\neq k}^{\infty} \frac{1}{z_k - z_j} \operatorname*{Res}_{z=z_j}\left[e^{-NG_N(z)}\right] \operatorname*{Res}_{z=z_k}\left[e^{-NG_N(z)}\right] M(z_j)\sigma_+ M(z_j)^{-1} M(z_k) \\ \times \sigma_+ M(z_k)^{-1} + \sum_{j=1}^{\infty} \operatorname*{Res}_{z=z_j}\left[e^{-NG_N(z)}\right] \operatorname*{Res}_{z=z_j}\left[\frac{e^{-NG_N(z)}}{z-z_j}\right] M(z_j)\sigma_+^2 M(z_j)^{-1}. \end{aligned} \tag{10.51}$$

Observe that the last sum is equal to 0, because $\sigma_+^2 = 0$. Furthermore, since $M(z_j) = M(+i0) + O(N^{-1})$, we obtain that

$$M(z_j)^{-1} M(z_k) = I + O(N^{-1}). \tag{10.52}$$

When we substitute I for $M(z_j)^{-1}M(z_k)$ in the first sum in (10.51), we again get 0. When we substitute $O(N^{-1})$ for $M(z_j)^{-1}M(z_k)$, we get a term of the order of $O(N^{-2\kappa_1})$. Thus,

$$R_1^{+,+} = O(N^{-2\kappa_1}). \tag{10.53}$$

Similarly,

$$R_1^{-,-} = O(N^{-2\kappa_1}). \tag{10.54}$$

Observe that by (10.21),

$$\kappa_1 = 1 + \frac{4\gamma}{\pi - 2\gamma} > 1. \tag{10.55}$$

Consider now

$$R_1^{-,+} = -\frac{1}{2\pi i}\int_{\Sigma_R^-} v_1^+(\lambda) J_R^\circ(\lambda)\, d\lambda. \tag{10.56}$$

When we substitute asymptotic expansion (10.49) into this formula and move the contour of integration, Σ_R^-, down, crossing the poles of $J_R^\circ(\lambda)$, we obtain the asymptotic expansion,

$$\begin{aligned} R_1^{-,+} \sim \sum_{j,k=1}^{\infty} \frac{1}{z_j + z_k} \operatorname*{Res}_{z=z_j}\left[e^{-NG_N(z)}\right] \operatorname*{Res}_{z=-z_k}\left[e^{NG_N(z)}\right] \\ \times M(z_j)\sigma_+ M(z_j)^{-1} M(-z_k)\sigma_+ M(-z_k)^{-1}. \end{aligned} \tag{10.57}$$

Since

$$M(z_j) = M(+i0) + O(N^{-1}), \qquad M(-z_k) = M(-i0) + O(N^{-1}), \tag{10.58}$$

we have that

$$\begin{aligned} &M(z_j)\sigma_+ M(z_j)^{-1} M(-z_k)\sigma_+ M(-z_k)^{-1} \\ &= M(+i0)\sigma_+ M(+i0)^{-1} M(-i0)\sigma_+ M(-i0)^{-1} + O(N^{-1}). \end{aligned} \tag{10.59}$$

From (7.56) we obtain, by a direct computation, that

$$\sigma_+ M(+i0)^{-1} M(-i0)\sigma_+ = -\sigma_+, \tag{10.60}$$

and from (10.20), (10.22), that

$$\operatorname*{Res}_{z=z_j}\left[e^{-NG_N(z)}\right]\operatorname*{Res}_{z=-z_k}\left[e^{NG_N(z)}\right]=C_jC_kN^{-\kappa_j-\kappa_k}(1+O(N^{-1})). \tag{10.61}$$

Hence

$$R_1^{-,+}\sim-\sum_{j,k=1}^{\infty}\frac{1}{z_j+z_k}N^{-\kappa_j-\kappa_k}\left[C_jC_kM(+i0)\sigma_+M(-i0)^{-1}+O(N^{-1})\right]. \tag{10.62}$$

Observe that

$$\sigma_+M(-i0)^{-1}M(+i0)\sigma_+=\sigma_+, \tag{10.63}$$

and, therefore, a similar computation for $R_1^{+,-}$ gives that

$$R_1^{+,-}\sim\sum_{j,k=1}^{\infty}\frac{1}{z_j+z_k}N^{-\kappa_j-\kappa_k}\left[C_jC_kM(-i0)\sigma_+M(+i0)^{-1}+O(N^{-1})\right]. \tag{10.64}$$

Since

$$M(-i0)\sigma_+M(+i0)^{-1}-M(+i0)\sigma_+M(-i0)^{-1}=I, \tag{10.65}$$

we obtain that

$$R_1^{+,-}+R_1^{-,+}\sim\sum_{j,k=1}^{\infty}\frac{1}{z_j+z_k}N^{-\kappa_j-\kappa_k}\left[C_jC_kI+O(N^{-1})\right]. \tag{10.66}$$

If we restrict this matrix formula to the elements (12) and (21), then we obtain that

$$(R_1^{+,-})_{12}+(R_1^{-,+})_{12}=O(N^{-2\kappa_1}),\qquad (R_1^{+,-})_{21}+(R_1^{-,+})_{21}=O(N^{-2\kappa_1}), \tag{10.67}$$

because $\frac{1}{z_j+z_k}=O(N)$. Finally, the cross terms of the form $R_1^{a,b}$, where $a=\pm$, $b=\alpha,\beta$, or vice versa, are estimated as

$$R_1^{a,b}=O(N^{-1-\kappa_1}),\qquad a=\pm,\ b=\alpha,\beta,\quad\text{or}\quad a=\alpha,\beta,\ b=\pm. \tag{10.68}$$

Summary for the quadratic term. By combining formulae (10.41), (10.46), (10.53), (10.54), (10.67), and (10.68), we obtain that

$$(R_1^{(2)})_{12}=(c_1)_{12}N^{-2}+O(N^{-2-\varepsilon}),\qquad (R_1^{(2)})_{21}=(c_1)_{21}N^{-2}+O(N^{-2-\varepsilon}), \tag{10.69}$$

where the matrix c_1 is given in (10.47) and $\varepsilon>0$.

Evaluation of the higher order terms. The higher order terms, $R_1^{(k)}$, $k\ge 3$, are evaluated in the same way as the quadratic terms, and we obtain that

$$(R_1^{(k)})_{12},(R_1^{(k)})_{21}=O(N^{-2-\varepsilon}),\qquad k\ge 3. \tag{10.70}$$

Consider, for instance, the cubic term,

$$R_1^{(3)}=-\frac{1}{2\pi i}\int_{\Sigma_R}v_2(\lambda)J_R^\circ(\lambda)d\lambda=\left(-\frac{1}{2\pi i}\right)^3\int_{\Sigma_R}\int_{\Sigma_R}\int_{\Sigma_R}\frac{J_R^\circ(\nu)J_R^\circ(\mu)J_R^\circ(\lambda)}{(\lambda_--\mu)(\mu_--\nu)}\,d\nu d\mu d\lambda. \tag{10.71}$$

As for the quadratic term, we split $R_1^{(3)}$ into a sum of terms $R_1^{a,b,c}$, and the only nontrivial terms in regard to estimate (10.70) are $R_1^{+,-,+}$ and $R_1^{-,+,-}$. We have that

$$R_1^{+,-,+}=\left(-\frac{1}{2\pi i}\right)^3\int_{\Sigma_R^+}\int_{\Sigma_R^-}\int_{\Sigma_R^+}\frac{J_R^\circ(\nu)J_R^\circ(\mu)J_R^\circ(\lambda)}{(\lambda_--\mu)(\mu_--\nu)}\,d\nu d\mu d\lambda. \tag{10.72}$$

We move the contour of integration Σ_R^+ up and the one Σ_R^- down, and obtain the asymptotic series,

$$\begin{aligned} R_1^{+,-,+} \sim \sum_{j,k,l=1}^{\infty} \frac{1}{(z_j+z_k)(z_k+z_l)} \operatorname*{Res}_{z=z_j}\left[e^{-NG_N(z)}\right] \operatorname*{Res}_{z=-z_k}\left[e^{NG_N(z)}\right] \operatorname*{Res}_{z=z_l}\left[e^{-NG_N(z)}\right] \\ \times M(z_j)\sigma_+ M(z_j)^{-1} M(-z_k)\sigma_+ M(-z_k)^{-1} M(z_l)\sigma_+ M(z_l)^{-1}. \end{aligned} \tag{10.73}$$

By using (10.58) and (10.60), we obtain that

$$R_1^{+,-,+} \sim i \sum_{j,k,l=1}^{\infty} \frac{1}{(z_j+z_k)(z_k+z_l)} N^{-\kappa_j-\kappa_k-\kappa_l}\left[C_jC_kC_l M(+i0)\sigma_+ M(-i0)^{-1} + O(N^{-1})\right]. \tag{10.74}$$

A similar computation for $R_1^{-,+,-}$ gives that

$$R_1^{-,+,-} \sim -i \sum_{j,k,l=1}^{\infty} \frac{1}{(z_j+z_k)(z_k+z_l)} N^{-\kappa_j-\kappa_k-\kappa_l}\left[C_jC_kC_l M(-i0)\sigma_+ M(+i0)^{-1} + O(N^{-1})\right], \tag{10.75}$$

and by using (10.65), we obtain that

$$R_1^{+,-,+} + R_1^{-,+,-} \sim -i \sum_{j,k,l=1}^{\infty} \frac{1}{(z_j+z_k)(z_k+z_l)} N^{-\kappa_j-\kappa_k-\kappa_l}\left[C_jC_kC_l I + O(N^{-1})\right], \tag{10.76}$$

hence

$$(R_1^{+,-,+})_{12} + (R_1^{-,+,-})_{12} = O(N^{-3\kappa_1+1}), \qquad (R_1^{+,-,+})_{21} + (R_1^{-,+,-})_{21} = O(N^{-3\kappa_1+1}). \tag{10.77}$$

Since $3\kappa_1 - 1 > 2$, we obtain estimate (10.70) for $R_1^{+,-,+} + R_1^{-,+,-}$. It is straightforward to get the estimate,

$$R_1^{a,b,c} = O(N^{-2-\varepsilon}), \tag{10.78}$$

for all other combinations of a, b, c and hence (10.70) follows. The same argument holds for $k > 3$.

Evaluation of R_{NN}. Let us go back now to formula (10.7) and evaluate the terms on the right in this formula with an error term of the order of $N^{-2-\varepsilon}$. From (5.10),

$$\beta_N - \alpha_N = \frac{2\pi}{\cos\frac{\pi\zeta}{2}} + N^{-2}\frac{2\gamma^2}{3(\pi-2\gamma)\cos\frac{\pi\zeta}{2}} + O(N^{-3}), \tag{10.79}$$

hence

$$\left(\frac{\beta_N-\alpha_N}{4}\right)^2 = \left(\frac{\pi}{2\cos\frac{\pi\zeta}{2}}\right)^2 + N^{-2}\frac{\pi\gamma^2}{6(\pi-2\gamma)\cos^2\frac{\pi\zeta}{2}} + O(N^{-3}), \tag{10.80}$$

Next, from (10.36), (10.68), and (10.70) we obtain that

$$\begin{aligned} \frac{\beta-\alpha}{4i}\left[(R_1)_{12} - (R_1)_{21}\right] &= \frac{\beta-\alpha}{2}\cos(N\omega) \sum_{j:\,\kappa_j\le 2} C_j N^{-\kappa_j} + cN^{-2} + O(N^{-2-\varepsilon}) \\ &= \cos(N\omega) \sum_{j:\,\kappa_j\le 2} c_j N^{-\kappa_j} + c^0 N^{-2} + O(N^{-2-\varepsilon}), \end{aligned} \tag{10.81}$$

where

$$c_j=\frac{\beta-\alpha}{2}C_j=\frac{2\gamma e^{\varphi(y_j)}}{\cos\frac{\pi\zeta}{2}}(-1)^j\sin\frac{\pi j}{1-\frac{2\gamma}{\pi}}\,, \tag{10.82}$$

and c^0 is a computable constant. From (10.7) we obtain now that

$$R_{NN}=\left(\frac{\pi}{2\cos\frac{\pi\zeta}{2}}\right)^2+\cos(N\omega)\sum_{j:\,\kappa_j\le 2}c_jN^{-\kappa_j}+cN^{-2}+O(N^{-2-\varepsilon})\,, \tag{10.83}$$

where

$$c=\frac{\pi\gamma^2}{6(\pi-2\gamma)\cos^2\frac{\pi\zeta}{2}}+c_0. \tag{10.84}$$

Here the first term in the expression for c comes from the difference $\left(\frac{\beta_N-\alpha_N}{4}\right)^2-\left(\frac{\beta-\alpha}{4}\right)^2$, see (10.80), while the second term, c_0, is determined by calculations of other terms of the order of N^{-2} on the right in formula (10.7). The constant c_0 can be evaluated explicitly by tracing down all the terms of the order of N^{-2} in the above computations. To avoid these somewhat tedious computations, we will use the fact that we know the exact expression for R_{NN} on the free fermion line.

Observe that c_0 is calculated in terms of contour integrals around the turning points α_N and β_N, and it depends only on the limiting values of the end points, α,β. The exact values of α,β are given in (4.5) and they depend on the parameter ζ only. This implies that c_0 is a function of the parameter ζ as well, $c_0=c_0(\zeta)$, and it is independent of γ. To find an exact value of $c_0(\zeta)$, consider the free fermion line $\gamma=\frac{\pi}{4}$. In this case $c=0$, which gives

$$c_0(\zeta)=-\frac{\pi^2}{48\cos^2\frac{\pi\zeta}{2}}\,. \tag{10.85}$$

Thus,

$$c=\frac{\pi\gamma^2}{6(\pi-2\gamma)\cos^2\frac{\pi\zeta}{2}}-\frac{\pi^2}{48\cos^2\frac{\pi\zeta}{2}}\,. \tag{10.86}$$

This proves formula (2.10) and hence Theorem 1.1.

11. Proof of Theorems 1.3 and 1.4

We omit the proof of Theorem 1.2, because it follows from Theorem 1.4.

Proof of Theorem 1.3. By (1.56) and (1.26),

$$\frac{\partial^2F_N}{\partial t^2}=\frac{R_N}{N^2}=\frac{1}{\gamma^2}\left[R+\cos(N\omega)\sum_{j:\,\kappa_j\le 2}c_jN^{-\kappa_j}+cN^{-2}+O(N^{-2-\varepsilon})\right]. \tag{11.1}$$

It is easy to check that

$$\frac{\partial^2F}{\partial t^2}=\frac{R}{\gamma^2}\,, \tag{11.2}$$

hence (1.57) follows. Theorem 1.3 is proved.

Proof of Theorem 1.4. By (1.26),

$$R_n=\frac{n^2R}{\gamma^2}e^{b_n}, \tag{11.3}$$

where

$$b_n = \cos(n\omega) \sum_{j:\, \kappa_j \le 2} d_j n^{-\kappa_j} - \kappa n^{-2} + O(n^{-2-\varepsilon}), \tag{11.4}$$

and $d_j = \frac{c_j}{R}$, $\kappa = -\frac{c}{R}$. From (1.27), (1.32) we obtain that

$$\kappa = -\frac{c}{R} = \frac{1}{12} - \frac{2\gamma^2}{3\pi(\pi - 2\gamma)}\,. \tag{11.5}$$

From (1.25) and (11.3) we obtain that

$$\tau_N = h_0^N \left(\frac{R}{\gamma^2}\right)^{\frac{N(N-1)}{2}} \left(\prod_{n=0}^{N-1} n!\right)^2 e^{B_N}, \tag{11.6}$$

where

$$B_N = (N-1)b_1 + (N-2)b_2 + \cdots + b_{N-1}, \tag{11.7}$$

hence by (1.34),

$$\begin{aligned} F_N = N^{-2} \ln \frac{\tau_N}{\left(\prod_{n=0}^{N-1} n!\right)^2} &= N^{-2}\left[N \ln h_0 + \frac{N(N-1)}{2} \ln \frac{R}{\gamma^2} + B_N\right] \\ &= \frac{1}{2} \ln \frac{R}{\gamma^2} + C_0 N^{-1} + N^{-2} B_N, \end{aligned} \tag{11.8}$$

where C_0 is a constant. Let us evaluate B_N. We have that

$$B_N = N(b_1 + b_2 + \cdots + b_N) - b_1 - 2b_2 - \cdots - N b_N, \tag{11.9}$$

and

$$b_1 + b_2 + \cdots + b_N = B - \sum_{n=N+1}^{\infty} b_n, \tag{11.10}$$

where

$$B = \sum_{n=1}^{\infty} b_n. \tag{11.11}$$

It follows from (11.4), that

$$\sum_{n=N+1}^{\infty} b_n = -\kappa N^{-1} + O(N^{-1-\varepsilon}), \tag{11.12}$$

because

$$\sum_{n=N+1}^{\infty} n^{-\kappa_j} \cos(n\omega) = O(N^{-\kappa_j}), \qquad 0 < \omega < 2\pi. \tag{11.13}$$

It also follows from (11.4), that

$$\sum_{n=1}^{N} n b_n = -\kappa \ln N + C_1 + O(N^{-\varepsilon}), \tag{11.14}$$

where C_1 is a constant, because

$$\sum_{n=1}^{N} n^{-\kappa_j+1}\cos(n\omega) = C(\kappa_j) + O(N^{-\kappa_j+1}), \qquad 0<\omega<2\pi. \tag{11.15}$$

Thus,

$$B_N = C_2 N + \kappa \ln N + C_3 + O(N^{-\varepsilon}), \tag{11.16}$$

where C_2, C_3 are some constants, hence from (11.8) we obtain that

$$F_N = F + c_0 N^{-1} + \kappa N^{-2}\ln N + C_3 N^{-2} + O(N^{-2-\varepsilon}), \tag{11.17}$$

where c_0 is a constant. This implies that

$$Z_N = C e^{N^2 f + N c_0} N^{\kappa}\left(1 + O(N^{-\varepsilon})\right), \tag{11.18}$$

where $C = e^{C_3}$. To finish the proof of Theorem 1.4, it remains to prove the following lemma.

Lemma 11.1. $c_0 = 0$.

Proof. By (2.6),

$$h_n = \left(\frac{n}{\gamma}\right)^{2n+1} h_{nn}, \tag{11.19}$$

and by (10.1),

$$h_{nn} = -2\pi i e^{n l_n} (S_1)_{12}. \tag{11.20}$$

Observe that by (5.67),

$$l_n = l + O(n^{-2}), \qquad l = 2\ln(\beta-\alpha) - 2 - 4\ln 2, \tag{11.21}$$

and by (10.5), (10.6),

$$(S_1)_{12} = (M_1)_{12} + (R_1)_{12} = -\frac{\beta-\alpha}{4i}\left(1 + O(n^{-1})\right). \tag{11.22}$$

Therefore,

$$h_n = \left(\frac{n}{\gamma}\right)^{2n+1} \frac{\pi(\beta-\alpha)}{2} \exp\left(nl + O(n^{-1})\right), \tag{11.23}$$

and

$$\begin{aligned}\tau_N = \prod_{n=0}^{N-1} h_n = h_0 \left(\prod_{n=1}^{N-1} n^{2n+1}\right) \gamma^{-N^2} \left(\frac{\pi(\beta-\alpha)}{2}\right)^{N-1} \\ \times \exp\left(\frac{N(N-1)}{2} l + O(\ln N)\right).\end{aligned} \tag{11.24}$$

By applying (11.21), (1.36), and (4.6), we obtain that

$$\tau_N = C_N \exp\left(N^2 F + O(\ln N)\right), \tag{11.25}$$

where C_N does not depend on γ and t. By (1.7) and (1.37), this implies that

$$Z_N = \tilde{C}_N \exp\left(N^2 f + O(\ln N)\right), \tag{11.26}$$

where $\tilde{C}_N$ also does not depend on γ and t. Since on the free fermion line, $Z_N = 1$ and $f = 0$, we obtain that

$$\ln \tilde{C}_N = O(\ln N), \tag{11.27}$$

hence

$$Z_N = \exp\left(N^2 f + O(\ln N)\right), \tag{11.28}$$

so that $c_0 = 0$. Lemma 11.1 is proved.

Appendix A. Large N asymptotics of $A(N)$ and $A(N;3)$

Large N asymptotics of $A(N)$. We will find in this appendix the large N asymptotics of

$$A(N) = \prod_{n=0}^{N-1} \frac{(3n+1)!n!}{(2n)!(2n+1)!}. \tag{A.1}$$

We start with the asymptotics of

$$a(N) = \prod_{n=1}^{N-1} n!. \tag{A.2}$$

We have that

$$\ln a(N) = \sum_{n=1}^{N} (N-n)\ln n = N^2 \sum_{n=1}^{N} \left(1 - \frac{n}{N}\right)\left(\ln \frac{n}{N}\right) N^{-1} + \sum_{n=1}^{N} (N-n)\ln N. \tag{A.3}$$

In addition,

$$\sum_{n=1}^{N} \left(1 - \frac{n}{N}\right)\left(\ln \frac{n}{N}\right) N^{-1} = -\frac{3}{4} + \frac{\ln N}{2N} + \frac{\ln(2\pi)}{2N} - \frac{\ln N}{12N^2} + \frac{\zeta'(-1)}{N^2} - \frac{1}{240N^4} + \dots, \tag{A.4}$$

where $\zeta(s)$ is the Riemann zeta-function. This gives

$$\ln a(N) = \frac{N^2 \ln N}{2} - \frac{3N^2}{4} + \frac{N\ln(2\pi)}{2} - \frac{\ln N}{12} + \zeta'(-1) - \frac{1}{240N^2} + \dots, \tag{A.5}$$

so that

$$a(N) = \prod_{n=1}^{N-1} n! = N^{\frac{N^2}{2}} e^{-\frac{3}{4}N^2} (2\pi)^{\frac{N}{2}} N^{-\frac{1}{12}} e^{\zeta'(-1) - \frac{1}{240N^2} + \cdots}. \tag{A.6}$$

Consider now

$$a_{31}(N) = \prod_{n=1}^{N-1} (3n+1)!. \tag{A.7}$$

We have that

$$\ln a_{31}(N) = b_1(N) + b_0(N) + b_{-1}(N) \tag{A.8}$$

where

$$b_j(N) = \sum_{n=1}^{N} (N-n)\ln(3n+j), \qquad j = 1, 0, -1. \tag{A.9}$$

Observe that

$$b_0(N) = \sum_{n=1}^{N} (N-n)\ln(3n) = \frac{(\ln 3)N(N-1)}{2} + \ln a(N), \tag{A.10}$$

hence by (A.5),

$$b_0(N)=\frac{(\ln 3)N(N-1)}{2}+\frac{N^2\ln N}{2}-\frac{3N^2}{4}+\frac{N\ln(2\pi)}{2}-\frac{\ln N}{12}+\zeta'(-1)-\frac{1}{240N^2}+\dots, \quad \text{(A.11)}$$

Now,

$$\begin{aligned} b_1(N)+b_{-1}(N)-2b_0(N)&=\sum_{n=1}^{N}(N-n)\ln\left(1-\frac{1}{9n^2}\right)\\ &=N\ln\left(\frac{3\sqrt{3}}{2\pi}\right)+\frac{\ln N}{9}+\gamma_0+\frac{2}{243N^2}+\dots, \end{aligned} \quad \text{(A.12)}$$

where γ_0 is a constant,

$$\gamma_0=\lim_{N\to\infty}\left[-\sum_{n=1}^{N}n\ln\left(1-\frac{1}{9n^2}\right)-\frac{\ln N}{9}\right]. \quad \text{(A.13)}$$

Therefore,

$$\begin{aligned} \ln a_{31}(N)=&\frac{3(\ln 3)N(N-1)}{2}+\frac{3N^2\ln N}{2}-\frac{9N^2}{4}+\frac{3N\ln(2\pi)}{2}-\frac{\ln N}{4}+3\zeta'(-1)\\ &-\frac{1}{80N^2}+\dots+N\ln\left(\frac{3\sqrt{3}}{2\pi}\right)+\frac{\ln N}{9}+\gamma_0+\frac{2}{243N^2}+\dots, \end{aligned} \quad \text{(A.14)}$$

and

$$a_{31}(N)=N^{\frac{3N^2}{2}}3^{\frac{3N^2}{2}}e^{-\frac{9}{4}N^2}(2\pi)^{\frac{N}{2}}N^{-\frac{5}{36}}e^{3\zeta'(-1)+\gamma_0-\frac{83}{19440N^2}+\dots}. \quad \text{(A.15)}$$

Finally,

$$\prod_{n=0}^{N-1}[(2n)!(2n+1)!]=\prod_{n=0}^{2N-1}n!=a(2N). \quad \text{(A.16)}$$

By (A.6),

$$a(2N)=(2N)^{2N^2}e^{-3N^2}(2\pi)^N(2N)^{-\frac{1}{12}}e^{\zeta'(-1)-\frac{1}{960N^2}+\dots}. \quad \text{(A.17)}$$

Thus, (A.1) reduces to

$$\begin{aligned} A(N)=\frac{a_{31}(N)a(N)}{a(2N)}&=\frac{N^{\frac{3N^2}{2}}3^{\frac{3N^2}{2}}e^{-\frac{9}{4}N^2}(2\pi)^{\frac{N}{2}}N^{-\frac{5}{36}}N^{\frac{N^2}{2}}e^{-\frac{3}{4}N^2}(2\pi)^{\frac{N}{2}}N^{-\frac{1}{12}}}{(2N)^{2N^2}e^{-3N^2}(2\pi)^N(2N)^{-\frac{1}{12}}}\\ &\times\frac{e^{3\zeta'(-1)+\gamma_0-\frac{83}{19440N^2}+\dots}e^{\zeta'(-1)-\frac{1}{240N^2}+\dots}}{e^{\zeta'(-1)-\frac{1}{960N^2}+\dots}}. \end{aligned} \quad \text{(A.18)}$$

By simplifying, we obtain that

$$A(N)=C\left(\frac{3\sqrt{3}}{4}\right)^{N^2}N^{-\frac{5}{36}}\left(1-\frac{115}{15552N^2}+O(N^{-3})\right), \quad \text{(A.19)}$$

where

$$C=2^{\frac{1}{12}}e^{3\zeta'(-1)+\gamma_0}. \quad \text{(A.20)}$$

Large N asymptotics of $A(N;3)$. From (1.48) we have that

$$\begin{cases} A(2m;3) = 3^{m^2} \dfrac{m!}{(3m)!} \displaystyle\prod_{k=0}^{m-1} \left[\frac{(3k+2)!}{(m+k)!}\right]^2, \\ A(2m+1;3) = 3^{m^2+m} \displaystyle\prod_{k=0}^{m-1} \left[\frac{(3k+2)!}{(m+k+1)!}\right]^2, \end{cases} \tag{A.21}$$

cf. [7]. Let us start with $A(2m;3)$. We can rewrite it as

$$A(2m;3) = 3^{m^2} \frac{m!}{(3m)!} \left[\frac{a_{32}(m)a(m)}{a(2m)}\right]^2, \tag{A.22}$$

where

$$a_{32}(m) = \prod_{k=0}^{m-1} (3k+2)!. \tag{A.23}$$

Observe that

$$a_{32}(m) = a_{31}(m) \prod_{k=0}^{m-1} (3k+2) = a_{31}(m) 3^m \frac{\Gamma\left(m+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}, \tag{A.24}$$

hence from (A.22) and (A.18) we obtain that

$$A(2m;3) = 3^{m^2} \frac{m!}{(3m)!} \left[\frac{3^m \Gamma\left(m+\frac{2}{3}\right) A(m)}{\Gamma\left(\frac{2}{3}\right)}\right]^2, \tag{A.25}$$

We have that

$$\frac{(m!)^3}{(3m)!} = 3^{-3m} \frac{2\pi m}{\sqrt{3}} e^{\frac{2}{9m}+O(m^{-3})} \tag{A.26}$$

and

$$\frac{\Gamma\left(m+\frac{2}{3}\right)}{m!} = m^{-\frac{1}{3}} e^{-\frac{1}{9m}+\frac{1}{162m^2}+\dots}. \tag{A.27}$$

By combining this with asymptotics (A.19), we obtain that

$$A(2m;3) = C_3 \left(\frac{3}{2}\right)^{4m^2} 3^{-m} (2m)^{\frac{1}{18}} \left(1 + \frac{77}{7776m^2} + O(m^{-3})\right), \tag{A.28}$$

where

$$C_3 = \frac{2^{\frac{10}{9}} \pi}{\left[\Gamma\left(\frac{2}{3}\right)\right]^2 \sqrt{3}} e^{6\zeta'(-1)+2\gamma_0}. \tag{A.29}$$

Consider now $A(2m+1;3)$. From (A.21),

$$A(2m+1;3) = 3^m \frac{(3m)!m!}{[(2m)!]^2} A(2m;3). \tag{A.30}$$

By using the Stirling formula we obtain that

$$3^m \frac{(3m)!m!}{[(2m)!]^2} = \left(\frac{3}{2}\right)^{4m} \frac{\sqrt{3}}{2} e^{\frac{1}{36m}+O(m^{-3})}. \tag{A.31}$$

Also,

$$\left(\frac{2m}{2m+1}\right)^{\frac{1}{18}} = e^{-\frac{1}{36m}+\frac{1}{144m^2}+O(m^{-3})} \tag{A.32}$$

By combining these formulae with (A.28), we get

$$A(2m+1;3) = C_3\left(\frac{3}{2}\right)^{(2m+1)^2}(\sqrt{3})^{-(2m+1)}(2m+1)^{\frac{1}{18}}\left(1+\frac{131}{7776m^2}+O(m^{-3})\right). \tag{A.33}$$

Appendix B. Proof of formula (4.10)

We have:

$$\int \omega(z)\,dz = z\omega(z) - \int z\omega'(z)dz. \tag{B.1}$$

From (4.3),

$$\begin{aligned}
\omega'(z) &= \frac{2}{i\pi}\left[\frac{\frac{\sqrt{\beta}}{2\sqrt{z-\alpha}} - \frac{i\sqrt{-\alpha}}{2\sqrt{z-\beta}}}{\sqrt{\beta(z-\alpha)} - i\sqrt{-\alpha(z-\beta)}} - \frac{1}{2z}\right] \\
&= \frac{1}{i\pi}\left[\frac{\left(\frac{\sqrt{\beta}}{\sqrt{z-\alpha}} - \frac{i\sqrt{-\alpha}}{\sqrt{z-\beta}}\right)\left(\sqrt{\beta(z-\alpha)} + i\sqrt{-\alpha(z-\beta)}\right)}{(\beta-\alpha)z} - \frac{1}{z}\right] \\
&= \frac{1}{\pi}\frac{\sqrt{\beta(-\alpha)}}{(\beta-\alpha)z}\left(\sqrt{\frac{z-\beta}{z-\alpha}} - \sqrt{\frac{z-\alpha}{z-\beta}}\right) = -\frac{\sqrt{\beta(-\alpha)}}{\pi z\sqrt{(z-\alpha)(z-\beta)}},
\end{aligned} \tag{B.2}$$

hence

$$\begin{aligned}
\int \omega(z)\,dz &= z\omega(z) + \frac{\sqrt{\beta(-\alpha)}}{\pi}\int \frac{dz}{\sqrt{(z-\alpha)(z-\beta)}} \\
&= z\omega(z) + \frac{2\sqrt{\beta(-\alpha)}}{\pi}\log\left(\sqrt{z-\alpha} + \sqrt{z-\beta}\right).
\end{aligned} \tag{B.3}$$

From (4.6), $\sqrt{\beta(-\alpha)} = \pi$, hence

$$g(z) = z\omega(z) + 2\log\left(\sqrt{z-\alpha} + \sqrt{z-\beta}\right) + C. \tag{B.4}$$

As $z \to \infty$,

$$g(z) = \log z + O(z^{-1}) = z[z^{-1} + O(z^{-2})] + 2[\log(2\sqrt{z}) + O(z^{-1})] + C, \tag{B.5}$$

hence $C = -1 - 2\ln 2$, and (4.10) follows.

Appendix C. Proof of Proposition 5.1

From (5.8), (5.9) we have that

$$F_N(\alpha_N, \beta_N) \equiv \frac{1}{2\pi}\int_{\alpha_N}^{\beta_N} \frac{V_N'(x)}{\sqrt{(x-\alpha_N)(\beta_N - x)}}dx = 0, \tag{C.1}$$

and

$$G_N(\alpha_N, \beta_N) \equiv \frac{1}{2\pi}\int_{\alpha_N}^{\beta_N} \frac{xV_N'(x)}{\sqrt{(x-\alpha_N)(\beta_N - x)}}dx = 1. \tag{C.2}$$

In (C.1), (C.2) we can rewrite the integrals as the contour integrals,

$$\begin{aligned} F_N(\alpha_N,\beta_N) &= \frac{1}{4\pi i}\oint_{\Gamma_\varepsilon}\frac{V_N'(z)}{\sqrt{(z-\alpha_N)(z-\beta_N)}}dz,\\ G_N(\alpha_N,\beta_N) &= \frac{1}{4\pi i}\oint_{\Gamma_\varepsilon}\frac{zV_N'(z)}{\sqrt{(z-\alpha_N)(z-\beta_N)}}dz, \end{aligned} \tag{C.3}$$

where the function $\sqrt{(z-\alpha_N)(z-\beta_N)}$ is considered on the principal sheet, with a cut on $[\alpha_N,\beta_N]$, and Γ_ε is a positively oriented contour on the complex plane around $[\alpha_N,\beta_N]$, which consists of the two circles, $\{|z-\alpha|=\varepsilon\}$ and $\{|z-\beta|=\varepsilon\}$, and the two intervals, $[\alpha+\varepsilon,\beta-\varepsilon]$, along the lower shore of the cut, and $[\beta-\varepsilon,\alpha+\varepsilon]$, along the upper shore, see Fig. 8. It follows from representation (C.3) that both F_N and G_N are analytic functions of α_N,β_N.

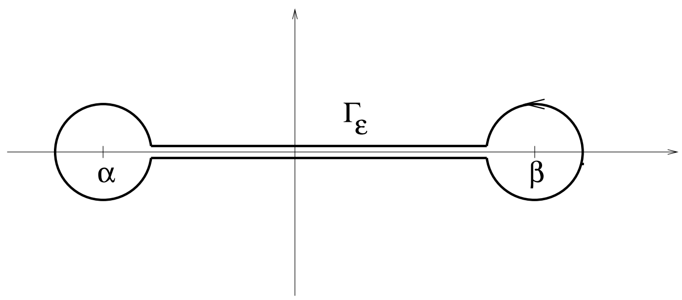

FIGURE 8. The contour Γ_ε.

By (5.13),

$$V_N'(z)=V'(z)+f(Nz), \tag{C.4}$$

where

$$V(z)=z\,\mathrm{sgn}\,\mathrm{Re}\,z-\zeta z,\qquad V'(z)=\mathrm{sgn}\,\mathrm{Re}\,z-\zeta, \tag{C.5}$$

and

$$f(z)=\frac{\pi}{2\gamma}\coth z\frac{\pi}{2\gamma}-\left(\frac{\pi}{2\gamma}-1\right)\coth z\left(\frac{\pi}{2\gamma}-1\right)-\mathrm{sgn}\,\mathrm{Re}\,z. \tag{C.6}$$

Therefore, we can rewrite equations (C.1), (C.2) as

$$\begin{cases} F(\alpha_N,\beta_N)\equiv\dfrac{1}{4\pi i}\displaystyle\oint_{\Gamma_\varepsilon}\frac{V'(z)}{\sqrt{(z-\alpha_N)(z-\beta_N)}}dz=-\dfrac{1}{4\pi i}\displaystyle\oint_{\Gamma_\varepsilon}\frac{f(Nz)}{\sqrt{(z-\alpha_N)(z-\beta_N)}}dz,\\ G(\alpha_N,\beta_N)\equiv\dfrac{1}{4\pi i}\displaystyle\oint_{\Gamma_\varepsilon}\frac{zV'(z)}{\sqrt{(z-\alpha_N)(z-\beta_N)}}dz=1-\dfrac{1}{4\pi i}\displaystyle\oint_{\Gamma_\varepsilon}\frac{zf(Nz)}{\sqrt{(z-\alpha_N)(z-\beta_N)}}dz. \end{cases} \tag{C.7}$$

We will assume that $\alpha_N-\alpha=O(N^{-2})$ and $\beta_N-\beta=O(N^{-2})$ as $N\to\infty$, where α and β solve the system

$$\begin{cases} F(\alpha,\beta)=0,\\ G(\alpha,\beta)=1, \end{cases} \tag{C.8}$$

and we will prove the existence of α_N,β_N by using the implicit function theorem. Observe that α and β are given by formulae (4.5).

The function $f(z)$ is exponentially decaying as $|\mathrm{Re}\, z|\to\infty$, and this allows us to evaluate the integrals on the right in (C.7) asymptotically, as $N\to\infty$. Namely,

$$\begin{aligned}\frac{1}{4\pi i}\oint_{\Gamma_\varepsilon}\frac{f(Nz)}{\sqrt{(z-\alpha_N)(z-\beta_N)}}dz&=\frac{1}{2\pi N}\int_{-N\alpha_N}^{N\beta_N}\frac{f(x)}{\sqrt{(N^{-1}x-\alpha_N)(\beta_N-N^{-1}x)}}dx\\&=\frac{1}{2\pi N\sqrt{(-\alpha_N)\beta_N}}\int_{-\infty}^{\infty}f(x)\left[1+\frac{x(\alpha_N+\beta_N)}{2N(-\alpha_N)\beta_N}\right]dx+O(N^{-3}).\end{aligned}\tag{C.9}$$

Observe that that $f(-x)=-f(x)$, hence

$$\int_{-\infty}^{\infty}f(x)dx=0\tag{C.10}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty}xf(x)dx&=2\int_0^\infty x\left(\frac{\pi}{\gamma}\frac{1}{e^{x\frac{\pi}{\gamma}}-1}-\left(\frac{\pi}{\gamma}-2\right)\frac{1}{e^{x(\frac{\pi}{\gamma}-2)}-1}\right)dx\\&=2\left(\frac{\gamma}{\pi}-\frac{\gamma}{\pi-2\gamma}\right)\int_0^\infty\frac{u}{e^u-1}du=-\frac{2\gamma^2\pi}{3(\pi-2\gamma)}.\end{aligned}\tag{C.11}$$

Also we can replace α_N,β_N for α,β in (C.9) and use formulae (4.6). This gives us that

$$\frac{1}{4\pi i}\oint_{\Gamma_\varepsilon}\frac{f(Nz)}{\sqrt{(z-\alpha_N)(z-\beta_N)}}dz=-N^{-2}\frac{\gamma^2\tan\frac{\pi\zeta}{2}}{3\pi^2(\pi-2\gamma)}+O(N^{-3})\,.\tag{C.12}$$

Similarly we obtain that

$$\frac{1}{4\pi i}\oint_{\Gamma_\varepsilon}\frac{zf(Nz)}{\sqrt{(z-\alpha_N)(z-\beta_N)}}dz=-N^{-2}\frac{\gamma^2}{3\pi(\pi-2\gamma)}+O(N^{-3})\,.\tag{C.13}$$

Thus, system (C.7) reduces to the following one:

$$\begin{cases}F(\alpha_N,\beta_N)=N^{-2}\dfrac{\gamma^2\tan\frac{\pi\zeta}{2}}{3\pi^2(\pi-2\gamma)}+O(N^{-3}),\\[2ex] G(\alpha_N,\beta_N)=1+N^{-2}\dfrac{\gamma^2}{3\pi(\pi-2\gamma)}+O(N^{-3}).\end{cases}\tag{C.14}$$

In the linear approximation the latter system reads

$$\begin{cases}(\alpha_N-\alpha)F_{\alpha_N}(\alpha,\beta)+(\beta_N-\beta)F_{\beta_N}(\alpha,\beta)=N^{-2}\dfrac{\gamma^2\tan\frac{\pi\zeta}{2}}{3\pi^2(\pi-2\gamma)}+O(N^{-3}),\\[2ex] (\alpha_N-\alpha)G_{\alpha_N}(\alpha,\beta)+(\beta_N-\beta)G_{\beta_N}(\alpha,\beta)=N^{-2}\dfrac{\gamma^2}{3\pi(\pi-2\gamma)}+O(N^{-3})\,.\end{cases}\tag{C.15}$$

The coefficients of this linear system can be evaluated explicitly. Namely, we have that

$$\begin{aligned}F(\alpha_N,\beta_N)&=-\frac{\zeta}{2}+\frac{1}{\pi}\arcsin\frac{\beta_N+\alpha_N}{\beta_N-\alpha_N}\,,\\G(\alpha_N,\beta_N)&=-\frac{\zeta(\beta_N+\alpha_N)}{4}+\frac{\sqrt{\beta_N(-\alpha_N)}}{\pi}+\frac{\beta_N+\alpha_N}{2\pi}\arcsin\frac{\beta_N+\alpha_N}{\beta_N-\alpha_N}\,,\end{aligned}\tag{C.16}$$

which gives that

$$\begin{aligned} &F_{\alpha_N}(\alpha,\beta)=\frac{1}{2\pi^2}\left(1+\sin\frac{\pi\zeta}{2}\right), \qquad F_{\beta_N}(\alpha,\beta)=\frac{1}{2\pi^2}\left(1-\sin\frac{\pi\zeta}{2}\right),\\ &G_{\alpha_N}(\alpha,\beta)=-\frac{1}{2\pi}\cos\frac{\pi\zeta}{2}, \qquad G_{\beta_N}(\alpha,\beta)=\frac{1}{2\pi}\cos\frac{\pi\zeta}{2}. \end{aligned} \tag{C.17}$$

By solving system (C.15), we obtain that

$$\begin{aligned} \alpha_N&=\alpha+N^{-2}\frac{\gamma^2\left(2\sin\frac{\pi\zeta}{2}-1\right)}{3(\pi-2\gamma)\cos\frac{\pi\zeta}{2}}+O(N^{-3}),\\ \beta_N&=\beta+N^{-2}\frac{\gamma^2\left(2\sin\frac{\pi\zeta}{2}+1\right)}{3(\pi-2\gamma)\cos\frac{\pi\zeta}{2}}+O(N^{-3}). \end{aligned} \tag{C.18}$$

The determinant of system (C.15) is not equal to zero, and this guarantees, by the implicit function theorem, that there exists a solution to (C.1), (C.2), which has the same asymptotics (C.18). Proposition 5.1 is proved.

Appendix D. Proof of Proposition 5.2

To prove (5.26), we would like to replace $r_N(\mu)$ and $r_N(x)$ in (5.16) by $r_N(0)$ and to estimate the error term as $O(N^{-2})$. Fix any $0<r<\frac{1}{2}\min\{-\alpha,\beta\}$.

Case 1, $\mu\in[\alpha+r,\beta-r]$. From (5.16) we have that

$$\begin{aligned} \rho_N^1(\mu)+\frac{1}{2\pi^2}k(N\mu)=-\frac{1}{2\pi^2}P.V.\int_{\alpha_N}^{\beta_N}\left[\sqrt{\frac{r_N(\mu)}{r_N(x)}}-1\right]\frac{f(Nx)dx}{\mu-x}\\ +\frac{1}{2\pi^2}\int_{\mathbb{R}^1\setminus[\alpha_N,\beta_N]}\frac{f(Nx)dx}{\mu-x}. \end{aligned} \tag{D.1}$$

Due to estimate (5.22), the second integral is exponentially small as $N\to\infty$, hence we can drop it. In the first integral we can drop the sign of the principal value, because the function under the integral is smooth, and we can restrict the limits of integration to $(\alpha+\frac{r}{2})$ and $(\beta-\frac{r}{2})$ plus an exponentially small term. Finally, the function

$$\left[\sqrt{\frac{r_N(\mu)}{r_N(x)}}-1\right]\frac{1}{\mu-x},$$

is a uniformly bounded analytic function in a fixed complex neighborhood of $(x,\mu)\in[\alpha+\frac{r}{2},\beta-\frac{r}{2}]\times[\alpha+r,\beta-r]$, hence

$$\int_{\alpha+\frac{r}{2}}^{\beta-\frac{r}{2}}\left[\sqrt{\frac{r_N(\mu)}{r_N(x)}}-1\right]\frac{f(Nx)dx}{\mu-x}=O(N^{-2}), \tag{D.2}$$

because f is an odd exponentially decaying function. This proves Proposition 5.2 for $\mu\in[\alpha+r,\beta-r]$.

Case 2, $\mu \in [\alpha_N, \beta_N] \setminus [\alpha + r, \beta - r]$. Suppose $\mu \in [\beta - r, \beta_N]$. From (5.16),

$$\begin{aligned}\rho^1_N(\mu) = &-\frac{\sqrt{r_N(\mu)}}{2\pi^2}\int_{\alpha_N}^{\alpha+2r}\frac{f(Nx)dx}{(\mu-x)\sqrt{r_N(x)}} - \frac{\sqrt{r_N(\mu)}}{2\pi^2}\int_{\alpha+2r}^{\beta-2r}\frac{f(Nx)dx}{(\mu-x)\sqrt{r_N(x)}}\\ &-\frac{\sqrt{r_N(\mu)}}{2\pi^2}P.V.\int_{\beta-2r}^{\beta_N}\frac{f(Nx)dx}{(\mu-x)\sqrt{r_N(x)}}\,.\end{aligned} \tag{D.3}$$

The first term is exponentially small as $N \to \infty$ (because f is exponentially decaying), and the second one is $O(N^{-2})$ (because f is odd and the integration is with respect to a smooth kernel). Let us consider the third term. We can rewrite it as

$$-\frac{\sqrt{r_N(\mu)}}{2\pi^2}P.V.\int_{\beta-2r}^{\beta_N}\frac{[f(Nx)-f(N\beta_N)]dx}{(\mu-x)\sqrt{r_N(x)}} - \frac{\sqrt{r_N(\mu)}}{2\pi^2}P.V.\int_{\beta-2r}^{\beta_N}\frac{f(N\beta_N)dx}{(\mu-x)\sqrt{r_N(x)}}\,. \tag{D.4}$$

The second term is evaluated explicitly as const.$f(N\beta_N)$, and it is exponentially small as $N \to \infty$. We can represent the first term as a half-sum of contour integrals over two contours, $\Gamma_\pm$, where Γ_+ (Γ_-) goes from $\beta - 2r$ to $\mu - \delta$, where $\delta = \frac{1}{3}(\beta_N - \mu)$, then along the upper (respectively, lower) semicircle of radius δ centered at μ, and then from $\mu + \delta$ to β_N. The both integrals are exponentially small as $N \to \infty$, hence the third term in (D.3) is exponentially small, and $\rho^1_N(\mu) = O(N^{-2})$ when $\mu \in [\beta - r, \beta_N]$. From (5.24) we obtain that $k(N\mu) = O(N^{-2})$ when $\mu \in [\beta - r, \beta_N]$. This proves (5.26) for $\mu \in [\beta - r, \beta_N]$. Similarly, it holds for $\mu \in [\alpha_N, \alpha + r]$. Proposition 5.2 is proved.

References

[1] R. Baxter, *Exactly solved models in statistical mechanics,* Academic Press, San Diego, CA.

[2] M.T. Batchelor, R.J. Baxter, M.J. O'Rourke, and C.M. Yung, Exact solution and interfacial tension of the six-vertex model with anti-periodic boundary conditions. *J. Phys. A* **28** (1995) 2759–2770.

[3] P. Bleher and A. Its, Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model. *Annals of Mathematics*, 1999, **150**, 185-266.

[4] P. Bleher and A. Its, Double scaling limit in the random matrix model: the Riemann-Hilbert approach. *Commun. Pure Appl. Math.*, **56** (2003), 433-516.

[5] P. Bleher and A. Its, Asymptotics of the partition function of a random matrix model. *Ann. Inst. Fourier* **55** (2005), 1943–2000.

[6] N.M. Bogoliubov, A.M. Kitaev, and M.B. Zvonarev, Boundary polarization in the six-vertex model, *Phys. Rev. E* **65** (2002), 026126.

[7] F. Colomo and A.G. Pronko, Square ice, alternating sign matrices, and classical orthogonal polynomials, *J. Stat. Mech. Theory Exp.* 2005, no. 1, 005, 33 pp. (electronic).

[8] P.A. Deift, T. Kriecherbauer, K.T-R. McLaughlin, New results on equilibrium measure for logarithmic potentials in the presence of an external field. *J. Approx. Theory* **95** (1998), 388-475.

[9] P.A. Deift, T. Kriecherbauer, K.T-R. McLaughlin, S. Venakides, and Z. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. *Commun. Pure Appl. Math.* **52** (1999), 1335-1425.

[10] N.M. Ercolani and K.T.-R. McLaughlin, Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques and applications to graphical enumeration, *Int. Math. Res. Not.*, **14** (2003), 755–820.

[11] A. G. Izergin, Partition function of the six-vertex model in a finite volume. *Sov. Phys. Dokl.* **32** (1987), 878.

[12] A. G. Izergin, D. A. Coker, and V. E. Korepin, Determinant formula for the six-vertex model. *J. Phys. A*, **25** (1992), 4315.

[13] V. E. Korepin, Calculation of norms of Bethe wave functions. *Commun. Math. Phys.* **86** (1982), 391-418.

[14] V. Korepin and P. Zinn-Justin, Thermodynamic limit of the six-vertex model with domain wall boundary conditions, *J. Phys. A* **33** No. 40 (2000), 7053
[15] T. Kriecherbauer, K. T-R. McLaughlin, Strong asymptotics of polynomials orthogonal with respect to Freud weights. *Int. Math. Res. Not.* **6** (1999), 299-333.
[16] G. Kuperberg, Another proof of the alternating sign matrix conjecture. *Int. Math. Res. Not.* (1996), 139-150.
[17] E. H. Lieb, Exact solution of the problem of the entropy of two-dimensional ice. *Phys. Rev. Lett.* **18** (1967) 692.
[18] E. H. Lieb, Exact solution of the two-dimensional Slater KDP model of an antiferroelectric. *Phys. Rev. Lett.* **18** (1967) 1046-1048.
[19] E. H. Lieb, Exact solution of the two-dimensional Slater KDP model of a ferroelectric. *Phys. Rev. Lett.* **19** (1967) 108-110.
[20] E. H. Lieb, Residual entropy of square ice. *Phys. Rev.* **162** (1967) 162.
[21] E. H. Lieb and F. Y. Wu, Two dimensional ferroelectric models, in *Phase Transitions and Critical Phenomena*, C. Domb and M. Green eds., vol. 1, Academic Press (1972) 331-490.
[22] W. H. Mills, D. P. Robbins, and H. Rumsey, Proof of the Macdonald conjecture. *Invent. Math.* **66** (1982) 73-87.
[23] W. H. Mills, D. P. Robbins, and H. Rumsey, Alternating-sign matrices and descending plane partitions. *J. Combin. Theory, Ser. A* **34** (1983) 340-359.
[24] K. Sogo, Toda molecule equation and quotient-difference method. *Journ. Phys. Soc. Japan* **62** (1993), 1887.
[25] G. Szego, *Orthogonal Polynomials.* Fourth edition. Colloquium Publications, vol. 23, AMS, Providence, RI, 1975.
[26] B. Sutherland, Exact solution of a two-dimensional model for hydrogen-bonded crystals. *Phys. Rev. Lett.* **19** (1967) 103-104.
[27] D. Zeilberger, Proof of the alternating sign matrix conjecture. *New York J. Math.* **2** (1996), 59-68.
[28] P. Zinn-Justin, Six-vertex model with domain wall boundary conditions and one-matrix model. *Phys. Rev. E* **62** (2000), 3411-3418.

Department of Mathematical Sciences, Indiana University-Purdue University Indianapolis, 402 N. Blackford St., Indianapolis, IN 46202, U.S.A.
E-mail address: bleher@math.iupui.edu

Department of Mathematical Sciences, Indiana University-Purdue University Indianapolis, 402 N. Blackford St., Indianapolis, IN 46202, U.S.A.
E-mail address: vvf@math.iupui.edu